Chapter 1

MULTI INDEX ASSIGNMENT PROBLEMS: COMPLEXITY, APPROXIMATION, APPLICATIONS

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1. INTRODUCTION

This chapter deals with approximation algorithms for and applications of multi index assignment problems (MIAPs). MIAPs and relatives of it have a relatively long history both in applications as well as in theoretical results, starting at least in the fifties (see e.g. Motzkin, 1952, Schell, 1955 and Koopmans and Beckmann, 1957). Here we intend to give the reader i) an idea of the range and diversity of practical problems that have been formulated as an MIAP, and ii) an overview on what is known on theoretical aspects of solving instances of MIAPs. In particular, we will discuss complexity and approximability issues for *special cases* of MIAPs. We feel that investigating special cases of MIAPs is an important topic since real-world instances almost always posses a certain structure that can be exploited when it comes to solving them.

Before doing so, let us first describe a somewhat frivolous, quite unrealistic situation that gives rise to an instance of an MIAP. Suppose that you have become mayor of a set of villages. Everything is running smoothly, except in one village where chaos seems to rule. You, as a responsible mayor should, start looking into this and after careful gathering of information you find that there 30 cats, 30 houses, 30 men and 30 women present in the village. Now, in order to establish some peace you impose that 30 "units" will have to be formed, each consisting of a cat, a house, a man, and a woman. It turns out that for each possible 4-tuple (and there are 30^4 =810000 of them), a number is known that reflects the happiness of this particular unit. Since you as a mayor want to maximize total happiness, the problem becomes to find those 30 units that achieve total happiness.

This is an example of a so-called axial 4-index assignment problem. Different versions of the MIAP may arise by varying the number of indices, or by asking not for 30 4-tuples but instead for 30^2 4-tuples such that each pair consisting of a cat and a house is represented exactly once in a 4-tuple, or by considering other objective functions (e.g. bottleneck).

This chapter is organized as follows. In Section 1.1 we give some technical preliminaries. Section 2. deals with the two versions of the 3-index assignment problem; Section 3. treats the general case. Finally, in Section 4. we discuss extensions of the MIAP.

1.1 TECHNICAL PRELIMINARIES

An introduction to the issue of approximation and complexity can be found in Papadimitriou, 1994 and Crescenzi and Kann, 1999. Here, we shortly describe two concepts we need.

- A *ρ-approximation algorithm* for a maximization (minimization) problem P is a polynomial time algorithm that, for all instances, outputs a solution with a value that is at least (at most) equal to ρ times the value of an optimal solution of P. Observe that for maximization problems $0 \leq \rho \leq 1$ and for minimization problems $\rho \geq 1$.

- A *polynomial time approximation scheme* (PTAS) for a maximization (minimization) problem is a family of polynomial time $(1-\epsilon)$-approximation algorithms ($(1+\epsilon)$-approximation algorithms) for all $\epsilon > 0$.

2. THE 3-INDEX ASSIGNMENT PROBLEM

In this section we introduce two types of 3-index assignment problems, the so-called axial 3-index assignment problem and the so-called planar 3-index assignment problem (these names were first used by Schell, 1955). For each of these types we give a formulation and discuss complexity issues. We mention heuristic and exact approaches that have appeared in literature and sketch applications.

2.1 THE AXIAL 3IAP

Given are 3 n-sets A_1, A_2 and A_3. For each triple in $A_1 \times A_2 \times A_3$ a number is known (either a profit w_{ijk} or a cost c_{ijk}). The problem is now to find n triples such that each element in $A_1 \cup A_2 \cup A_3$ is in exactly one triple. So in the axial 3IAP one is asked to output n triples that either maximize total profit or minimize total cost.

Maximization.

$$
\begin{aligned}
(A3IAPMAX) \quad \text{maximize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_{ijk}x_{ijk} \\
\text{subject to} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} \le 1 \text{ for } k = 1,\ldots,n, \\
& \sum_{i=1}^{n}\sum_{k=1}^{n} x_{ijk} \le 1 \text{ for } j = 1,\ldots,n, \\
& \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} \le 1 \text{ for } i = 1,\ldots,n, \\
& x_{ijk} \in \{0,1\} \text{ for } i,j,k = 1,\ldots,n.
\end{aligned}
$$

Notice that this formulation generalizes the well-known (2-index) assignment problem; moreover in this generalization we lose the integrality property of the assignment problem. In other words, when relaxing the integrality constraints and solving the resulting Linear Programming formulation, fractional values for the variables may arise.

The axial 3IAP is a generalization of the 3-dimensional matching problem (3DM). Indeed: for $w_{ijk} \in \{0,1\}$ 3DM is *equivalent* to 3IAP. Thus, concerning complexity and approximation for the 3IAP with coefficients in $\{0,1\}$, we can state the following (see Crescenzi and Kann, 1999):

- It appears in the seminal work of Karp, 1972, and thus has the honor of being one of the seven first problems to be proved to be NP-complete.

- Subsequently, Kann, 1991 proves that even for a special case called *bounded* 3DM there is no PTAS (unless P=NP). Bounded means that for each element in $A_1 \cup A_2 \cup A_3$, the number of triples with weight 1 in which an element of $A_1 \cup A_2 \cup A_3$ occurs is bounded by a prespecified constant $B \ge 3$.

- When the coefficients are *planar* (admittedly, this may be confusing terminology) the problem remains NP-hard (Dyer and Frieze, 1983), but a PTAS exists (Nishizeki and Chiba, 1988). Planar means that the graph constructed as follows: have a node for each element of $A_1 \cup A_2 \cup A_3$ and for each triple with weight 1, and connect two nodes iff the corresponding triple contains the corresponding element, is planar.

- Hurkens and Schrijver, 1989 give a $\frac{2}{3} - \epsilon$ approximation algorithm (see Section 3.).

When allowing arbitrary values for the coefficients the following is known:

- Hausmann et al., 1980 show in a more general context that a simple greedy algorithm achieves a performance guarantee of $\frac{1}{3}$ (see Section 3.).
- Arkin and Hassin, 1998 give a $\frac{1}{2} - \epsilon$ approximation algorithm.
- if there exist numbers a_i, b_j and c_k, $i, j, k = 1, \ldots, n$ such that $w_{ijk} = a_i \cdot b_j \cdot c_k$, for all i, j, k, the problem is easy (Gilbert and Hofstra, 1988, Burkard et al., 1996).
- A recent result in Barvinok et al., 1998 is as follows. Let the $3n$ points be points in R^l for some dimension l, and let distances d be computed according to some geometric norm. For the case of polyhedral norms (which includes the rectilinear norm and the sup norm) and with $w_{ijk} = d_{ij} + d_{ik} + d_{jk}$ for all i, j, k (see Section 3. for a general framework encompassing this cost-structure) it is claimed that the problem is polynomially solvable.

Minimization.

$$\begin{array}{lll} (A3IAPMIN) & \text{minimize} & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk}x_{ijk} \\ & \text{subject to} & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} = 1 \text{ for } k = 1, \ldots, n, \\ & & \sum_{i=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \text{ for } j = 1, \ldots, n, \\ & & \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \text{ for } i = 1, \ldots, n, \\ & & x_{ijk} \in \{0,1\} \text{ for } i, j, k = 1, \ldots, n. \end{array}$$

Concerning complexity and approximation we have the following:

- Again, the NP-hardness of this problem follows from Karp, 1972.
- Even stronger, it is proven in Crama and Spieksma, 1992 that no polynomial time algorithm can even achieve a constant performance ratio unless P=NP. In fact, this result already holds in each of the following two cases. Let d_{ij}, d_{ik} and d_{jk} be nonnegative numbers specified for each pair of elements in different sets. One can interpret these numbers as distances (not necessarily in a metric space). The result mentioned above holds when
  - $c_{ijk} = d_{ij} + d_{ik} + d_{jk}$ for all i, j, k, and when

- $c_{ijk} = \min\{d_{ij} + d_{ik}, d_{ij} + d_{jk}, d_{ik} + d_{jk}\}$ for all i, j, k.

  However, when in addition to one of these cost-structures the d's satisfy the triangle inequality a $\frac{4}{3}$ approximation algorithm exists.

- If the $3n$ points are located in the plane and distances are computed according to the Euclidean norm, Spieksma and Woeginger, 1996 prove that minimizing the total circumference ($c_{ijk} = d_{ij} + d_{ik} + d_{jk}$ for all i, j, k) as well as minimizing the total area is NP-hard.

- Also, if $c_{ijk} = a_i \cdot b_j \cdot c_k$, for all i, j, k the problem remains NP-hard and even hard to approximate (Burkard et al., 1996), which contrasts with the maximization case.

Applications and solution methods. Pierskalla, 1967; Pierskalla, 1968 mention a number of settings in which axial 3IAPs arise: capital investment, dynamic facility location, satellite launching. Further, in Crama et al., 1996 a situation in the assembly of printed circuit boards is described that is modeled using formulation (A3IAPMIN). Arbib et al., 1999 describe a problem in perishable production planning formulated as an axial 3IAP with $\{0, 1\}$ coefficients.

Exact approaches for axial 3IAPs are presented in Leue, 1972, Hansen and Kaufman, 1973, Burkard and Fröhlich, 1980 and in Balas and Saltzman, 1991. Burkard and Rudolf, 1993 study the effect of different branching rules in a branch and bound framework. A GRASP for the axial 3IAP is presented in Lidstrom et al., 1999.

The polytope induced by the convex hull of the feasible solutions to (A3IAPMIN) is studied in Balas and Saltzman, 1989, Balas and Qi, 1993, Qi et al., 1994 and Gwan and Qi, 1992.

2.2 THE PLANAR 3IAP

Given are 3 n-sets A_1, A_2 and A_3. For each triple in $A_1 \times A_2 \times A_3$ a number p_{ijk} is known. The problem is now to find n^2 triples such that each pair of elements from $(A_1 \times A_2) \cup (A_1 \times A_3) \cup (A_2 \times A_3)$ is in exactly one triple. So in the planar 3IAP one is asked to output n^2 triples containing each pair of indices exactly once that either maximize or minimize the sum of the p_{ijk}'s corresponding to selected triples.

$$(P3IAP) \quad \text{maximize/minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk} x_{ijk}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ijk} = 1 \text{ for } j, k = 1, \ldots, n,$$

$$\sum_{j=1}^{n} x_{ijk} = 1 \text{ for } i,k = 1,\ldots,n,$$
$$\sum_{k=1}^{n} x_{ijk} = 1 \text{ for } i,j = 1,\ldots,n,$$
$$x_{ijk} \in \{0,1\} \text{ for } i,j,k = 1,\ldots,n.$$

The planar 3IAP is proved to be NP-complete in Frieze, 1983 even when the $p_{ijk} \in \{0,1\}$. When the coefficients are invariant with respect to one index, that is they satisfy $p_{ijk} = p_{ijl}$ for all $k \neq l$, Gilbert and Hofstra, 1987 show that the problem becomes polynomially solvable.

Applications and solution methods. Planar 3IAPs find applications in timetabling (see e.g. Hilton, 1980). Gilbert and Hofstra, 1987 formulate a practical rostering problem as a planar 3IAP. In Balas and Landweer, 1983 a problem in launching satellites is modeled using a planar 3IAP with a special objective function.

Branch and bound methods for planar 3IAP are proposed by Vlach, 1967, Burkard and Fröhlich, 1980 and by Magos and Miliotis, 1994. A heuristic procedure is proposed in Evans, 1981. A tabu search approach is presented in Magos, 1996.

The polytope coresponding to the convex hull of feasible solutions of (P3IAP) has been studied in Euler et al., 1986.

3. MIAPS

Of course, one can straightforwardly generalize the formulation (A3IAP) and (P3IAP) to higher dimensions using multiple summations. Here, however we give a more compact formulation (see also Queyranne and Spieksma, 1999). Observe that in case of a k index assignment problem there are $k-1$ variants possible (cf. the 3 index case). Thus, in order to specify a k index assignment problem another parameter, say q, needs to be given. We refer to the resulting problem as a q-fold kIAP.

Given are k n-sets $A_1, A_2, \ldots, A_k$, and let $A = \bigotimes_{i=1}^{k} A_i = A_1 \times A_2 \times \ldots \times A_k$. Thus, A is the set of all k-tuples $a = (a(1), a(2), \ldots, a(k)) \in A$. There is a variable x_a for each $a \in A$. Given costs c_a for all $a \in A$ we can formulate the objective function as minimize $\sum_{a \in A} c_a x_a$.

Given an integer q, $1 \leq q \leq k-1$, the constraints of a q-fold k index assignment problem are defined as follows. Let Q be the set of all subsets of $\{1, 2, \ldots, k\}$ consisting of $k-q$ elements. An element F of this set Q corresponds to a set of "fixed" indices. Given such an F, let $A_F = \bigotimes_{f \in F} A_f$. Further, given some $g \in A_F$, let $A(F,g) = \{a \in A |\ a(f) = g(f)$ for all $f \in F\}$ be the set of k-tuples that coincide with g on the fixed indices.

$$(q\text{-fold } kIAP) \quad \text{max/min} \quad \sum_{a \in A} c_a x_a$$
$$\text{subject to} \quad \sum_{a \in A(F,g)} x_a = 1 \text{ for all } g \in A_F, F \in Q,$$
$$x_a \in \{0,1\} \text{ for all } a \in A.$$

Observe that the case $q = 1$ corresponds to a planar k index assignment problem, whereas the case $q = k - 1$ corresponds to an axial problem. Some of the approaches mentioned for the 3 index case can be generalized to the k index case:

- The results in Hausmann et al., 1980 imply a $\frac{1}{k}$ approximation algorithm for the axial k index assignment problem (when maximizing).
- The approach in Hurkens and Schrijver, 1989 implies a $\frac{2}{k} - \epsilon$ approximation algorithm for the axial k index assignment problem with $\{0,1\}$ cost coefficients (when maximizing).

Further, Kuipers, 1990 proves for the axial kIAP with $\{0,1\}$ coefficients (when maximizing) that the ratio between the value of the LP-relaxation and an optimal solution is bounded by $k-1$, $k \geq 2$.

An important special case of the (min) axial multi index assignment problem is the case where the cost coefficients are so-called *decomposable* (see Bandelt et al., 1994). Decomposable costs arise when a number d_{st} is specified for each pair of elements $(s,t) \in A_i \times A_j$, $(i, j = 1, \ldots, k)$ and a function h exists such that the costs of a k-tuple depend on the d's involved in the k-tuple. More formally, let $c_a = h(d_{a(1),a(2)}, d_{a(1),a(3)}, \ldots, d_{a(k-1),a(k)})$ for all $a \in A$. Different possibilities exist for specifying h; popular examples are sum costs, tour costs, diameter costs and others (see Bandelt et al., 1994). Depending upon the specific form of the function h there are results in approximation: for instance there is a $2 - \frac{2}{k}$ approximation algorithm for sum costs (see Bandelt et al., 1994).

Applications and solution methods. An important application that is modeled using an axial multi index assignment problem occurs in the field of target tracking and plays a central role in operating surveillance systems. A description is as follows. Consider a radar system used to monitor some area. At each of discrete time-units $t_1, t_2, \ldots$ a set of *measurements* (called a *scan*) induced by targets becomes available. For simplicity it is usually assumed that each target moves in a straight line in the plane with a constant velocity (but other movement patterns are conceivable). The problem is now to associate the measurements with the targets, or in other words to identify *tracks*, each consisting of a series of measurements (at most 1 from each scan) caused by the same target. Such a track reflects the trajectory of the target. This so-called *data-association* problem can be naturally formulated as an axial multi index assignment problem by defining the sets A_i as the set of measurements in scan i. The cost-coefficients then express how likely it is that a particular set of measurement (one from each scan) are caused by the same target.

Solution methods for this application based on Lagrangian relaxation are proposed in Pattipatti et al., 1992, Poore, 1994, Poore and Rijavec, 1993. Other approaches are described in Murphey et al., 1998.

An application concerning routing in meshes that can be formulated as an axial MIAP is described in Fortin and Tusera, 1994.

4. EXTENSIONS

As in the 2-dimensional case, an important extension of multi index assignment problems is to multi index *transportation* problems. Observe that in the formulations considered sofar all right-hand sides are 1. By relaxing this property (and by allowing sets A_i of possibly unequal size) a more general problem arises which we call multi index (integer) transportation problems (depending upon whether integrality requirements are present). See Queyranne and Spieksma, 1999 for an overview of these problems.

MIAPs as discussed in this chapter can be extended in various other ways:

- The bottleneck case. Axial 3IAPs with a bottleneck objective are considered in Malhotra et al., 1985 and Geetha and Vartak, 1994. Special cases of axial 3IAPs with a bottleneck objective function are considered in Klinz and Woeginger, 1996.

- Multi criteria. An solution method for an axial 3IAP with two criteria can be found in Geetha and Vartak, 1989.

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