$1, \frac{3}{2}, \frac{13}{7}, \ldots$

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1 Prelude

Monday, February 1, 1988. This day was the first day of the first course for Ph. D. students organized by the LNMB; it was also the first day of my "AIO"-ship at the Department of Mathematics at Maastricht University. Hence, one could say that a lustrum for the LNMB marks at the same time a lustrum for myself as a researcher. Perhaps because of this I feel somewhat attached to the LNMB and am grateful for this opportunity to contribute to the lustrum book.

2 Introduction

The standard framework of worst-case analysis is usually as follows: imagine you're given this large instance (say $\mathcal{I}$) of a difficult (= $\mathcal{NP}$-hard) problem (say a minimization problem) with not much time to solve it. So using some guidelines you think reasonable (the simple heuristic H) you construct a feasible solution ($H(\mathcal{I})$) with a certain cost ($c(H(\mathcal{I}))$. And then, in some cases, one can produce the following statement:

$$c(H(\mathcal{I})) \leq \alpha \cdot \mathrm{OPT}(\mathcal{I}) \quad \text{for some } \alpha \in \mathbb{R}, \textit{ for all } \mathcal{I}, \tag{1}$$

where $\mathrm{OPT}(\mathcal{I})$ denotes the optimal value associated to instance $\mathcal{I}$.

At very first sight, it may seem that magic was needed to produce such a statement: how else can one say "for all $\mathcal{I}$" without enumerating all instances and running the heuristic on it? Well, when you get down to it, magic is perhaps not the right word describing how such a statement can be produced, but it sometimes feels that way when writing it down.

Let me spend a few words concerning terminology. The validity of (1) implies that α is an *upper bound for the worst-case ratio* of the heuristic H with respect to the problem considered. When, in addition to this, an instance $\mathcal{I}$ can be exhibited for which the heuristic actually delivers a solution with value $\alpha \cdot OPT(\mathcal{I})$, α can be called the *worst-case ratio* (or the *ratio* for short).

Not surprisingly, this contribution deals with the worst-case analysis of a simple heuristic for a difficult problem. In Section 3 I describe a certain k-dimensional assignment problem and a heuristic, and I discuss the results known sofar (which are succinctly summarized in the title). Section 4 introduces a basic observation, which is used in Section 5 to deduce, for the special case $k = 5$, two inequalities and an LP-model. This model enables us to construct, for the case $k = 5$, a tighter upper bound for the ratio than currently known. Section 6 indicates that this approach can be generalized to arbitrary k.

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3 A difficult problem, a simple heuristic, and the results known sofar

3.1 A difficult problem

Let $k \geq 2$ be an integer, and let $K = \{1, \ldots, k\}$. Given is a complete k-partite graph $G = (V = \cup_{l=1}^{k} V_l, E)$ with $|V_l| = n$ for all $l \in K$. I will sometimes refer to the vertices of a set V_l as vertices of color l. For each $u \in V_i, v \in V_j, i, j \in K, i \neq j$, there is a nonnegative cost $c_{uv} \in \mathbb{R}$ associated to the corresponding edge $\{u, v\} \in E$. These costs are not arbitrary: apart from being nonnegative, I assume that the so-called triangle inequality holds, that is:

$$c_{uv} + c_{vw} \geq c_{uw} \text{ for all } u \in V_i, v \in V_j, w \in V_l, \text{ and for all } i, j, l \in K, i \neq j, i \neq l, j \neq l. \quad (2)$$

Define a clique $C \subset V$ as a set of vertices such that $|C \cap V_l| = 1$ for all $l \in K$. The cost of a clique is defined as $\sum_{u,v \in C} c_{uv}$. The problem is now to find a partition of V into n disjoint cliques $C_1, \ldots, C_n$ such that the sum of the costs of the cliques is minimal. This problem is called the k-dimensional assignment problem with clique costs (k-DAPC). Notice that for $k = 2$ the problem boils down to an assignment problem.

3.2 A simple heuristic

Given a large instance of this problem with not much time to solve it, what can one do? An idea is the following: specify a sequence of the k colors, and repeatedly assign the vertices of color i to the sets of vertices consisting of vertices of colors $1, \ldots, i-1$, for $i = 2, \ldots, k$. The total cost of assigning a particular vertex of color i to a particular set of vertices of colors $1, \ldots, i-1$ is taken to be the sum of the costs of the $i-1$ edges between the vertex of color i on the one hand, and each of the other $i-1$ vertices on the other hand. So the algorithm consists of solving iteratively $n-1$ assignment problems.

In order to describe this algorithm more formally, let $V_l = \{v_{l1}, \ldots, v_{ln}\}$ for all $l \in K$. Here is the heuristic H^k:

Step 1: Choose a sequence, say $1, 2, \ldots, k$.

Step 2: Set $P_j = \{v_{1j}\}$ for $j = 1, \ldots, n$. Set $l = 1$.

Step 3: While $l < k$ do

- i: For all $i, j = 1, \ldots, n$, compute $\delta_{ij} = \sum_{w \in P_j} c_{w, v_{l+1,i}}$.
- ii: Find an optimal assignment A between the vertices of V_{l+1} and the partial cliques P_j, $j = 1, \ldots, n$ with respect to the cost function δ.
- iii: Extend the partial cliques P_j according to the assignment, that is for each $\{P_j, v_{l+1,i}\} \in A$ set $P_j := P_j \cup \{v_{l+1,i}\}$.
- iv: $l := l + 1$.

Step 4: The cliques are now formed by P_j for $j = 1, \ldots, n$. Stop.

3.3 The results known sofar

Two chapters of my thesis (Spieksma [3]) are devoted to (variants of) k-DAPC: Chapter 3, co-authored with Yves Crama (published as [2]) shows that 3-DAPC is $\mathcal{NP}$-hard, presents the heuristic H^3 and proves the second number in the title as the ratio for this heuristic applied to 3-DAPC. Chapter 4 of my thesis, co-authored with Hans-Jürgen Bandelt and Yves Crama (published as [1]) presents the heuristic H^k of Subsection 3.2, proves that $\frac{1}{2}k$ is an upper bound for the ratio, (or in other words shows that:

$$c(H^k) \leq \frac{1}{2}k \cdot OPT \text{ for all } k \geq 2), \tag{3}$$

and proves that $\frac{13}{7}$ is the ratio for 4-DAPC. For $k \geq 5$ it is unknown what the worst-case ratio of H^k with respect to k-DAPC is.

Let me add here that in [1] a heuristic is presented of polynomial complexity (albeit with a larger complexity than H^k) which achieves a ratio of $2 - \frac{2}{k}$ for all $k \geq 2$, thus a heuristic with a worst-case ratio bounded by 2 for all $k \geq 2$.

Finally, the reader may wonder whether there are practical applications which motivate the study of this heuristic for k-DAPC with $k \geq 5$. The answer is no; not that I'm aware of. The best reason I can give for looking into this is curiosity concerning the series $1, \frac{3}{2}, \frac{13}{7}, \ldots$ Where does it go, and is there perhaps a closed formula which describes these ratios? Perhaps disappointingly, these questions will remain unanswered at the end of this contribution.

4 An observation

The following notation will be used. Given an instance of k-DAPC, let F denote an optimal solution, and let H denote a solution found by H^k. Define for all $i, j \in K, i < j$:

$$F_{ij} := \{\{u,v\} \mid u \in V_i,\ v \in V_j,\ u \text{ and } v \text{ are contained in a clique from } F\},$$

$$d_{ij}^F := \sum_{\{u,v\} \in F_{ij}} c_{uv},$$

$$H_{ij} := \{\{u,v\} \mid u \in V_i,\ v \in V_j,\ u \text{ and } v \text{ are contained in a clique from } H\}, \text{ and}$$

$$d_{ij}^H := \sum_{\{u,v\} \in H_{ij}} c_{uv}.$$

More generally, given a solution S to k-DAPC, let for all $i, j \in K, i < j$:

$$S_{ij} := \{\{u,v\} \mid u \in V_i,\ v \in V_j,\ u \text{ and } v \text{ are contained in a clique from } S\}, \text{ and}$$

$$d_{ij}^S := \sum_{\{u,v\} \in S_{ij}} c_{uv}.$$

Also, I use:

$$d_{\bullet,i} := \sum_{j=1}^{i-1} d_{j,i} \quad \text{for all } i = 2, \ldots, k.$$

A crucial observation for the analysis which follows is the following one. Consider solution H found by the heuristic H^k. Given H, let us construct an alternative solution as follows. Choose a color $i, i < k$. Reassign the vertices of V_k to the partial cliques according to how in an *optimal solution* to this instance of k-DAPC the vertices of V_k and V_i are matched together. Call this solution S. Due to the fact that the heuristic finds an optimal matching between V_k and the partial cliques, we have the following inequality:

$$c(H^k) = d^H_{\bullet,2} + d^H_{\bullet,3} + \ldots + d^H_{\bullet,k} \leq d^H_{\bullet,2} + d^H_{\bullet,3} + \ldots + d^H_{\bullet,k-1} + d^S_{\bullet,k}.$$

Since $d^S_{ik} = d^F_{ik}$ by construction, and since $d^S_{jk} \leq d^H_{ji} + d^F_{ik}$ for all $j \neq i$ by the triangle inequalities (2), we obtain the following observation:

Observation:

$$c(H^k) \leq d^H_{\bullet,2} + d^H_{\bullet,3} + \ldots + d^H_{\bullet,k-1} + d^H_{1,i} + d^H_{2,i} + \ldots + d^H_{i,k-1} + (k-1)d^F_{ik}, \tag{4}$$

for all $i = 1, \ldots, k-1$.

5 The case $k = 5$

This section consists of three subsections. In each of the first two, an inequality is deduced. In the final subsection I use these inequalities in an LP-model to obtain a tighter upperbound than predicted by (3) for the case $k = 5$.

5.1 Inequality 1

Lemma 1 $c(H^5) \leq 4d^F_{12} + 3(d^F_{13} + d^F_{23}) + 6d^F_{34} + 4d^F_{45}$.

Proof:

We can derive this inequality as follows. Obviously:

$$c(H^5) = d^H_{\bullet,2} + d^H_{\bullet,3} + d^H_{\bullet,4} + d^H_{\bullet,5}. \tag{5}$$

Now consider iteratively each of the following inequalities, substitute it in (5), and proceed:

1: $d^H_{\bullet,5} \leq d^H_{\bullet,4} + 4d^F_{45}$.

This follows from Inequality (4) with $k = 5$ and $i = 4$.

2: $2d^H_{\bullet,4} \leq 2d^H_{\bullet,3} + 6d^F_{34}$.

This follows from Inequality (4) with $k = 4$ and $i = 3$.

3: $\frac{3}{2}d^H_{\bullet,3} \leq \frac{3}{2}d^H_{\bullet,2} + 3d^F_{23}$.

This follows from Inequality (4) with $k = 3$ and $i = 2$.

4: $\frac{3}{2}d^H_{\bullet,3} \leq \frac{3}{2}d^H_{\bullet,2} + 3d^F_{13}$.

This follows from Inequaltity (4) with $k = 3$ and $i = 1$.

5: $4d^H_{\bullet,2} \leq 4d^F_{\bullet,2}$.

Trivial.

□

5.2 Inequality 2

Lemma 2 $$c(H^5) \le \frac{46}{9} d^F_{12} + \frac{23}{9}(d^F_{13} + d^F_{23}) + \frac{4}{3}(d^F_{14} + d^F_{24} + d^F_{34}) + \frac{4}{3}(d^F_{15} + d^F_{25} + d^F_{35}).$$

Proof:
As before we can derive this inequality as follows. Obviously:

$$c(H^5) = d^H_{\bullet,2} + d^H_{\bullet,3} + d^H_{\bullet,4} + d^H_{\bullet,5}. \tag{6}$$

Now consider iteratively each of the following inequalities, substitute it in (6), and proceed:

1: $$d^H_{\bullet,5} \le \frac{4}{3}(d^F_{15} + d^F_{25} + d^F_{35}) + \frac{1}{3} d^H_{\bullet,4} + \frac{2}{3}(d^H_{\bullet,3} + d^H_{\bullet,2}).$$

This follows by summing Inequality (4) with $k = 5$ for $i = 1, 2, 3$, and dividing it by 3.

2: $$\frac{4}{3} d^H_{\bullet,4} \le \frac{4}{3} d^F_{\bullet,4} + \frac{8}{9}(d^H_{\bullet,3} + d^H_{\bullet,2}).$$

This follows by summing Inequality (4) with $k = 4$ for $i = 1, 2, 3$, and dividing it by 3.

3: $$\frac{23}{9} d^H_{\bullet,3} \le \frac{23}{9}(d^F_{\bullet,2} + d^F_{\bullet,3}).$$

This follows by summing Inequality (4) with $k = 3$ for $i = 1$ and 2, and dividing it by 2.

4: $$\frac{23}{9} d^H_{\bullet,2} \le \frac{23}{9} d^F_{\bullet,2}.$$

Trivial.

□

5.3 An LP-model

How to use the two inequalities from Lemma's 1 and 2 from the previous subsections to obtain a better upperbound for the ratio than 5/2 as predicted by (3)? I am going to construct a linear combination of these two inequalities and next, using the triangle inequalities $d^F_{ij} \le d^F_{il} + d^F_{lj}$ for $i, j, l = 1, \dots, 5$ (which hold due to inequalities (2)), I intend to minimize the largest coeffficient of some d^F_{ij} term. Obviously, the largest coefficient in the resulting inequality determines an upper bound for the ratio. The problem of minimizing the largest coefficient can be casted into an LP-framework in the following way. Consider the following decision variables:

- z_{ij}: coefficient of d^F_{ij} in resulting inequality; $i, j = 1, \dots, 5,\ i < j$,
- x_{ijl}: coefficient of triangle inequality $d^F_{ij} \le d^F_{il} + d^F_{lj}$ for $i, j, l = 1, \dots, 5,\ i < j, l \ne i, l \ne j$,
- α_1: coefficient of inequality of Lemma 1, and
- α_2: coefficient of inequality of Lemma 2.

Here is the LP-model called LP-5-DAPC:

$(LP-5-DAPC)$ minimize w

subject to

$$z_{12} = 4\alpha_1 + \frac{46}{9}\alpha_2 - x_{123} - x_{124} - x_{125} + x_{132} + x_{142} + x_{152} + x_{231} + x_{241} + x_{251}, \tag{7}$$

$$z_{13} = 3\alpha_1 + \frac{23}{9}\alpha_2 - x_{132} - x_{134} - x_{135} + x_{123} + x_{143} + x_{153} + x_{231} + x_{341} + x_{351}, \tag{8}$$

$$z_{14} = \frac{4}{3}\alpha_2 - x_{142} - x_{143} - x_{145} + x_{124} + x_{134} + x_{154} + x_{241} + x_{341} + x_{451}, \tag{9}$$

$$z_{15} = \frac{4}{3}\alpha_2 - x_{152} - x_{153} - x_{154} + x_{125} + x_{135} + x_{145} + x_{251} + x_{351} + x_{451}, \tag{10}$$

$$z_{23} = 3\alpha_1 + \frac{23}{9}\alpha_2 - x_{231} - x_{234} - x_{235} + x_{123} + x_{132} + x_{243} + x_{253} + x_{342} + x_{352}, \tag{11}$$

$$z_{24} = \frac{4}{3}\alpha_2 - x_{241} - x_{243} - x_{245} + x_{124} + x_{142} + x_{234} + x_{254} + x_{342} + x_{452}, \tag{12}$$

$$z_{25} = \frac{4}{3}\alpha_2 - x_{251} - x_{253} - x_{254} + x_{125} + x_{152} + x_{235} + x_{245} + x_{352} + x_{452}, \tag{13}$$

$$z_{34} = 6\alpha_1 + \frac{4}{3}\alpha_2 - x_{341} - x_{342} - x_{345} + x_{134} + x_{143} + x_{234} + x_{243} + x_{354} + x_{453}, \tag{14}$$

$$z_{35} = \frac{4}{3}\alpha_2 - x_{351} - x_{352} - x_{354} + x_{135} + x_{153} + x_{235} + x_{253} + x_{345} + x_{453}, \tag{15}$$

$$z_{45} = 4\alpha_1 - x_{451} - x_{452} - x_{453} + x_{145} + x_{154} + x_{245} + x_{254} + x_{345} + x_{354}, \tag{16}$$

$w \geq z_{ij}$ for all $i, j = 1, \ldots, 5, i < j$,

$\alpha_1 + \alpha_2 = 1$, and

all variables ≥ 0.

An explanation of the constraints of LP-5-DAPC is as follows. Consider equality (7). In this equality the coefficient of d^F_{12} in the final inequality (z_{12}) is determined. Now, obviously z_{12} must be equal to the linear combination of the inequalities of Lemma's 1 and 2 restricted to the term d^F_{12} $(4\alpha_1 + \frac{46}{9}\alpha_2)$ plus a term which indicates the "usage" of triangle inequalities in which d^F_{12} occurs. There are three triangle inequalities with d^F_{12} appearing on the left-hand side, namely $d^F_{12} \leq d^F_{13} + d^F_{23}$, $d^F_{12} \leq d^F_{14} + d^F_{24}$ and $d^F_{12} \leq d^F_{15} + d^F_{25}$, so the corresponding x-variables have coefficient -1, and there are six triangle inequalities with d^F_{12} appearing on the right-hand side (this is easily verified), so the corresponding x-variables have coefficient 1. In fact, each of the equalities (8)-(16) can be explained in a similar way. The remaining constraints of LP-5-DPAC are straightforward.

Solving this model yields $w = 2.301471$ and hence the following statement can be produced (cf. (1))

$$c(H^5) \leq 2.301471 \cdot \mathrm{OPT}(\mathcal{I}) \quad \text{for all } \mathcal{I}.$$

Notice that the upperbound for the ratio of H^5 with respect to 5-DAPC has decreased from 2.5 (see Inequality (3)) to 2.301471.

6 Discussion

My approach has focused exclusively on 5-DAPC. In order to generalize this approach to other values of k, one should be able to generalize the inequalities deduced in Lemma's 1 and 2. It turns out that this is possible, however, it seems out of the scope of this contribution to state these inequalities and prove their validity for general k. However, I couldn't resist constructing the corresponding LP's for $k = 6$ and 7 and solve them. It turned out that in each case there was an improvement (compared to (3)) of the upperbound for the ratio. So here is a table with the current best upper bounds for ratio's of heuristic H^k for k-DAPC for $k = 5, 6$ and 7.

k-DAPC	Upper bound for ratio
$k = 5$	2.301471
$k = 6$	2.699225
$k = 7$	3.062519

References

[1] Bandelt, H.-J., Y. Crama and F.C.R. Spieksma (1994), "Approximation algorithms for multi-dimensional assignment problems with decomposable costs", Discrete Applied Mathematics **49**, 25-50.

[2] Crama, Y. and F.C.R. Spieksma (1992), "Approximation algorithms for three- dimensional assignment problems with triangle inequalities", European Journal of Operational Research **60**, 273-279.

[3] Spieksma, F.C.R. (1992), "Assignment and scheduling algorithms in automated manufacturing", Ph. D. thesis of Maastricht University.