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APPROXIMATION ALGORITHMS FOR RECTANGLE STABBING AND INTERVAL STABBING PROBLEMS*

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Abstract. In the weighted rectangle stabbing problem we are given a grid in $\mathbb{R}^2$ consisting of columns and rows each having a positive integral weight, and a set of closed axis-parallel rectangles each having a positive integral demand. The rectangles are placed arbitrarily in the grid with the only assumption being that each rectangle is intersected by at least one column or row. The objective is to find a minimum-weight (multi)set of columns and rows of the grid so that for each rectangle the total multiplicity of selected columns and rows stabbing it is at least its demand. A special case of this problem, called the interval stabbing problem, arises when each rectangle is intersected by exactly one row. We describe an algorithm called *STAB*, which is shown to be a constant-factor approximation algorithm for different variants of this stabbing problem.

Key words. rectangle stabbing, approximation algorithms, combinatorial optimization

AMS subject classifications. 68W25, 68R05, 90C27

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1. Introduction.

The *weighted rectangle stabbing problem* (WRSP) can be described as follows: given are a grid in $\mathbb{R}^2$ consisting of columns and rows each having a positive integral weight, and a set of closed axis-parallel rectangles each having a positive integral demand. The rectangles are placed arbitrarily in the grid with the only assumption being that each rectangle is intersected by at least one column or row. The objective is to find a minimum-weight (multi)set of columns and rows of the grid so that for each rectangle the total multiplicity of selected columns and rows stabbing this rectangle equals at least its demand. (A column or row is said to stab a rectangle if it intersects it.)

A special case of the WRSP is the case where each rectangle is intersected by exactly one row; we will refer to the resulting problem as the *weighted interval stabbing problem* (WISP), or ISP in the case of unit weights (see Figure 1 for an example of an instance of the ISP).

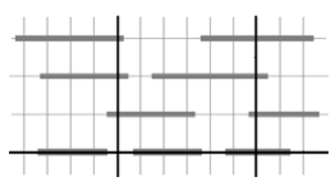

FIG. 1. *An instance of ISP with unit demands. The rectangles (or intervals in this case) are in grey; the columns and row in black constitute a feasible solution.*

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Motivation. Although at first sight the WRSP may seem rather specific, it is not difficult to see that the following two problems can be reduced to WRSP.

- *Solving special integer programming problems.* The following type of integer linear programming problem can be reformulated as instances of WRSP: $minimize\{wx|\ (B|C)x \geq b,\ x \in \mathbb{Z}^l\}$, where B and C are both 0,1-matrices with consecutive 1's in the rows (so-called interval matrices; see, e.g., Schrijver [11]), $b \in \mathbb{Z}^n_+$, $w \in \mathbb{Z}^l_+$. Indeed, construct a grid which has a column for each column in B and a row for each column in C. For each row i of matrix $B|C$, draw a rectangle i such that it intersects only the columns and rows of the grid corresponding to the positions of 1's in row i. Observe that this construction is possible since B and C have consecutive 1's in the rows. To complete the construction, assign demand b_i to each rectangle i and a corresponding weight w_j to each column and row of the grid. Let the decision variables x describe the multiplicities of the columns and rows of the grid. In this way we have obtained an instance of WRSP. In other words, integer programming problems where the columns of the constraint matrix A can be permuted such that $A = (B|C)$, with B and C each being an interval matrix, are special cases of WRSP.
- *Stabbing geometric figures in the plane.* Given a set of arbitrary connected closed geometric sets in the plane, use a minimum number of straight lines of two given directions to stab each of these sets at least once. Indeed, by introducing a new coordinate system specified by the two directions and by replacing each closed connected set by a closed rectangle defined by the projections of the set to the new coordinate axes, we obtain an instance of the problem of stabbing rectangles using a minimum number of axis-parallel lines. More specifically, we define a grid whose rows and columns are axis-parallel lines containing the rectangles' edges. We can restrict attention to those lines since any axis-parallel line stabbing some set of rectangles can be replaced by a line stabbing this set and containing a rectangle's edge. Therefore, the problem of stabbing the rectangles with axis-parallel lines reduces to the problem of stabbing them with the rows and columns of the grid.

Literature. The WRSP and its special case WISP have already received attention in the literature. Motivated by an application in parallel processing, Gaur, Ibaraki, and Krishnamurti [3] present a 2-approximation algorithm for the WRSP with unit weights and demands, which admits an easy generalization to arbitrary weights and demands. Furthermore, Hassin and Megiddo [4] (mentioning military and medical applications) study a number of special cases of the problem of stabbing geometric figures in $\mathbb{R}^2$ by a minimum number of straight lines. In particular, they present a 2-approximation algorithm for the task of stabbing connected figures of the same shape and size with horizontal and vertical lines. Moreover, they study the case of stabbing horizontal line segments of length K, whose endpoints have integral x-coordinates, with a minimum number of horizontal and vertical lines, and give a $2 - \frac{1}{K}$-approximation algorithm for this problem. In our setting this corresponds to the ISP with unit demands, where each rectangle in the input is intersected by exactly K columns. Finally, Călinescu et al. [2], mentioning applications in embedded sensor networks, show that the problem of separating n points in the plane with a minimum number of axis-parallel lines is a special case of the unweighted rectangle stabbing problem.

Concerning computational complexity, a special case of ISP where each rectangle is stabbed by at most two columns is shown to be APX-hard in [9].

Our results. We present here an approximation algorithm called *STAB* for different variants of WISP (see, e.g., Vazirani [12] for an overview on approximation algorithms). First, we show that *STAB* is a $\frac{1}{(1-(1-1/k)^k)}$-approximation algorithm for ISP_k, the variant of ISP where each row intersects at most k rectangles (e.g., the instance depicted in Figure 1 is an instance of ISP_3). Observe that *STAB* is a $\frac{4}{3}$-approximation algorithm for the case $k = 2$, and that *STAB* is an $\frac{e}{e-1}$-approximation algorithm for the case where the number of rectangles sharing a row is unlimited ($k = \infty$). Thus, *STAB* improves upon the results described in Hassin and Megiddo [4] (for $K \geq 3$) and does not impose any restrictions on the number of columns intersecting rectangles. Second, we show that *STAB* is an $\frac{e}{e-1}$-approximation algorithm for the *weighted* case of ISP_∞, i.e., the case where the columns and the rows of the grid have arbitrary positive integral weights. Third, we state here that the algorithm described by Gaur, Ibaraki, and Krishnamurti [3] can be generalized to yield a $\frac{q+1}{q}$-approximation algorithm for WRSP where the demand of each rectangle is bounded from below by an integer q. Observe that this provides a 2-approximation algorithm for the WRSP described in the introduction, where $q = 1$. Thus, this is an improvement upon the approximation ratio of the algorithm of Gaur, Ibaraki, and Krishnamurti [3] for instances with a lower bound on the rectangles' demands that is larger than 1. For the proof of this result, we refer to Kovaleva [7].

Our algorithms are based on rounding the linear programming relaxation of an integer programming formulation in an interesting way. We use the following property present in our formulation: The variables can be partitioned into two sets such that when the values of one set are fixed, one can compute the optimal values of the other variables in polynomial time, and vice versa. Next, we consider different ways of rounding one set of variables and compute each time the optimal values of the remaining variables, while keeping the best solution.

We also show that there exist instances of ISP_2 and ISP_∞ (see section 3) and WRSP (see [7]) for which the ratio between the values of a natural integer linear programming (ILP) formulation and its linear programming relaxation (LP-relaxation) is equal (or arbitrarily close) to the obtained approximation ratios. This suggests that these approximation ratios are unlikely to be improved by an LP-rounding algorithm based on the natural ILP formulation.

2. Preliminaries. Let us formalize the definition of WRSP. Let the grid in the input consist of t columns and m rows, numbered consecutively from left to right and from bottom to top, with positive weight w_c (v_r) attached to each column c (row r). Further, we are given n rectangles such that rectangle i has demand $d_i \in \mathbb{Z}_+$ and is specified by leftmost column l_i, rightmost column r_i, top row t_i, and bottom row b_i.

Let us give a natural ILP formulation of WRSP. In this paper we use notation $[a : b]$ for the set of integers $\{a, a+1, \ldots, b\}$. The decision variables $y_c, z_r \in \mathbb{Z}_+$, $c \in [1 : t]$, $r \in [1 : m]$, denote the multiplicities of column c and row r, respectively.

$$\text{(1)} \qquad \text{Minimize} \quad \sum_{r=1}^{m} v_r z_r + \sum_{c=1}^{t} w_c y_c$$

$$\text{(2)} \qquad \text{subject to} \quad \sum_{r \in [b_i:t_i]} z_r + \sum_{c \in [l_i:r_i]} y_c \geq d_i \quad \forall i \in [1:n],$$

$$\text{(3)} \qquad z_r, y_c \in \mathbb{Z}_+^1 \qquad \forall r, c.$$

In a vector notation this can be represented as

$$\text{Minimize} \quad vz + wy \tag{4}$$
$$\text{subject to} \quad Bz + Cy \geq d, \tag{5}$$
$$z \in \mathbb{Z}_+^m,\ y \in \mathbb{Z}_+^t, \tag{6}$$

where $B \in \{0,1\}^{n\times m}$ and $C \in \{0,1\}^{n\times t}$ are the constraint matrices of inequalities (2). The linear programming relaxation is obtained by replacing the integrality constraints (6) by the nonnegativity constraints $z \in \mathbb{R}_+^m,\ y \in \mathbb{R}_+^t$.

For an instance $\mathcal{I}$ of WRSP and a vector $a \in \mathbb{Z}^n$, we introduce two auxiliary ILP problems:

$$\text{IP}^z(\mathcal{I},a)\text{:} \quad (7) \qquad \begin{array}{ll} \text{Minimize} & vz \\ \text{subject to} & Bz \geq a, \\ & z \in \mathbb{Z}_+^m. \end{array}$$

$$\text{IP}^y(\mathcal{I},a)\text{:} \quad (8) \qquad \begin{array}{ll} \text{Minimize} & wy \\ \text{subject to} & Cy \geq a, \\ & y \in \mathbb{Z}_+^t. \end{array}$$

LEMMA 2.1. *For any $a \in \mathbb{Z}^n$, the LP-relaxation of each of the problems IP$^z(\mathcal{I},a)$ and IP$^y(\mathcal{I},a)$ is integral.*

Proof. As was previously observed in [3], matrices B and C have a so-called consecutive 1's property. This implies that these matrices are totally unimodular (see, e.g., Schrijver [11]), which implies the lemma. □

COROLLARY 2.2. *The optimum value of IP$^z(\mathcal{I},a)$ (IP$^y(\mathcal{I},a)$) is smaller than or equal to the value of any feasible solution to its LP-relaxation.*

COROLLARY 2.3. *The problem IP$^z(\mathcal{I},a)$ (IP$^y(\mathcal{I},a)$) can be solved in polynomial time. Its optimal solution coincides with that of its LP-relaxation.*

In fact, the special structure of IP$^z(\mathcal{I},a)$ and IP$^y(\mathcal{I},a)$ allows us to solve it via a minimum cost flow algorithm: Let $MCF(p,q)$ denote the time needed to solve the minimum cost flow problem on a network with p nodes and q arcs. A proof of the following lemma can also be found in Veinott and Wagner [13].

LEMMA 2.4. *The problem IP$^z(\mathcal{I},a)$ (IP$^y(\mathcal{I},a)$) can be solved in time $O(MCF(t,n+t))$ ($O(MCF(m,n+m))$).*

Proof. Consider the LP-relaxation of formulation IP$^y(\mathcal{I},a)$ and substitute the current variables by new variables $u_0,\ldots,u_t$ as $y_c = u_c - u_{c-1}\ \forall c \in [1:t]$. Then it transforms into

$$\begin{array}{ll} \text{Minimize} & -w_1u_0 + (w_1 - w_2)u_2 + \cdots + (w_{t-1} - w_t)u_{t-1} + w_tu_t \\ \text{subject to} & u_{r_i} - u_{l_i-1} \geq a_i \quad \forall i \in [1:n], \\ & u_c - u_{c-1} \geq 0 \quad \forall c \in [1:t]. \end{array} \tag{9}$$

Let us denote the vector of objective coefficients, the vector of right-hand sides, and the constraint matrix by w, a, and C, respectively, and the vector of variables by u. Then (8) can be represented as $\{\text{minimize } wu \mid Cu \geq a\}$. Its dual is $\{\text{maximize } ax \mid C^Tx = w, x \geq 0\}$. Observe that this is a minimum cost flow formulation with flow conservation constraints $C^Tx = w$, since C^T has exactly one 1 and one -1 in each column. Given an optimal solution to the minimum cost flow problem, one can obtain the optimal dual solution $u_0,\ldots,u_t$ via a shortest path computation (see Ahuja, Magnanti, and Orlin [1]), and thus optimal $y_1,\ldots,y_t$ values as well. □

3. Algorithm *STAB*. Recall that the interval stabbing problem WISP refers to the restriction of WRSP, where each rectangle in the input is intersected by exactly one row. We also refer by WISP_k to WISP, where each row intersects at most k rectangles. We assume in this section that all demands are unit ($d_i = 1$, $i \in [1:n]$), thus resulting in the following formulation:

$$\text{Minimize} \quad \sum_{r=1}^{m} v_r z_r + \sum_{c=1}^{t} w_c y_c \tag{10}$$

$$\text{subject to} \quad z_{\rho_i} + \sum_{c \in [l_i : r_i]} y_c \geq 1 \quad \forall i \in [1:n], \tag{11}$$

$$z_r, y_c \in \mathbb{Z}_+^1 \qquad \forall r, c. \tag{12}$$

Here we denote by ρ_i the index of the row intersecting rectangle i.

First we describe algorithm *STAB* for WISP. In subsection 3.1 we show that it achieves a ratio of $\frac{1}{1-(1-1/k)^k}$ for the unweighted version of WISP_k: ISP_k. In subsection 3.2 we prove that *STAB* achieves a ratio of $\frac{e}{e-1}$ for WISP. Subsection 3.3 shows that the integrality gap between the values of a natural integer programming formulation of ISP_k and its LP-relaxation for $k = 2$ and $k = \infty$ coincides with the approximation ratio of the algorithm. An alternative algorithm for the case $k = 2$ yielding the same worst-case ratio (i.e., $\frac{4}{3}$) is described in Kovaleva and Spieksma [8].

Informally, algorithm *STAB* can be described as follows: Solve the LP-relaxation of (10)–(12), and denote the solution found by $(y^{\text{lp}}, z^{\text{lp}})$. Assume, without loss of generality, that the rows are sorted as $z_1^{\text{lp}} \geq z_2^{\text{lp}} \geq \cdots \geq z_m^{\text{lp}}$. At each iteration j ($j = 0, \ldots, m$) we solve the problem (10)–(12) with a fixed vector z, the first j elements of which are set to 1, and the others to 0. As shown in Lemma 2.4, this can be done in polynomial time using a minimum cost flow algorithm. Finally, we take the best of the resulting $m+1$ solutions. A formal description of *STAB* is shown in Figure 2.

We use notation $value(y,z) \equiv \sum_{c=1}^{t} y_c + \sum_{r=1}^{m} z_r$, $value(y) \equiv \sum_{c=1}^{t} y_c$, and $value(z) \equiv \sum_{r=1}^{m} z_r$.

1. solve the LP-relaxation of (10)–(12), and obtain its optimal solution $(y^{\text{lp}}, z^{\text{lp}})$;
2. reindex the rows of the grid so that $z_1^{\text{lp}} \geq z_2^{\text{lp}} \geq \cdots \geq z_m^{\text{lp}}$;
3. $V \leftarrow \infty$;
4. for $j = 0$ to m
   - for $i = 1$ to j $\bar{z}_i \leftarrow 1$,
   - for $i = j+1$ to m $\bar{z}_i \leftarrow 0$.
   - solve $\text{IP}^y(\mathcal{I}, b)$, where $b_i = 1 - \bar{z}_{\rho_i}$, $\forall i \in [1:n]$, and obtain $\bar{y}$;
   - if $value(\bar{y}, \bar{z}) < V$, then $V \leftarrow value(\bar{y}, \bar{z})$, $y^* \leftarrow \bar{y}$, $z^* \leftarrow \bar{z}$;
5. return (y^*, z^*).

FIG. 2. *Algorithm STAB.*

3.1. The approximation result for ISP_k. In this subsection we show that algorithm *STAB* is a $\frac{1}{1-(1-1/k)^k}$-approximation algorithm for ISP_k. Let us first adapt the ILP formulation (10)–(12) to ISP_k with unit demands:

$$\text{(13)} \qquad \text{Minimize} \quad \sum_{c=1}^{t} y_c + \sum_{r=1}^{m} z_r$$

$$\text{(14)} \qquad \text{subject to} \quad z_{\rho_i} + \sum_{c \in [l_i : r_i]} y_c \geq 1 \quad \forall i \in [1:n],$$

$$\text{(15)} \qquad z_r, y_c \in \mathbb{Z}_+ \qquad \forall r, c.$$

THEOREM 3.1. *Algorithm STAB is a* $\frac{1}{1-(1-1/k)^k}$*-approximation algorithm for* ISP_k.

Proof. Consider an instance $\mathcal{I}$ of ISP_k, and let $(y^{\mathrm{lp}}, z^{\mathrm{lp}})$ and (y^*, z^*) be, respectively, an optimal LP solution and the solution returned by the algorithm for $\mathcal{I}$. We prove the theorem by establishing that

$$\text{(16)} \qquad value(y^*, z^*) \leq \frac{1}{1-(1-1/k)^k}\, value(y^{\mathrm{lp}}, z^{\mathrm{lp}}).$$

It is enough to prove the result for instances satisfying the following assumption: We assume that the optimal LP solution satisfies constraints (14) at equality; i.e.,

$$\text{(17)} \qquad z^{\mathrm{lp}}_{\rho_i} + \sum_{c \in (l_i : r_i)} y^{\mathrm{lp}}_c = 1 \quad \forall i \in [1:n].$$

We now sketch why we can assume that (17) holds. Indeed, suppose that (17) does not hold for some intervals i of some instance $\mathcal{I}$. Then we modify $\mathcal{I}$ by shortening those intervals for which (17) does not hold. More precisely, by splitting the columns with y^{lp}-values we shorten the appropriate intervals so that the assumption becomes true (see Figure 3 for an example). Thus, given $\mathcal{I}$ and $(y^{\mathrm{lp}}, z^{\mathrm{lp}})$, we create an instance $\mathcal{I}'$ for which (17) holds. It is now easy to check that an optimal LP solution for $\mathcal{I}$ (with the split columns) is also an optimal LP solution for $\mathcal{I}'$. Since in $\mathcal{I}'$ the intervals have become shorter, algorithm *STAB* applied to $\mathcal{I}'$ returns a solution with a value equal to or larger than the value of the solution returned for $\mathcal{I}$. Then inequality (16) proven for $\mathcal{I}'$ implies this inequality for $\mathcal{I}$ as well.

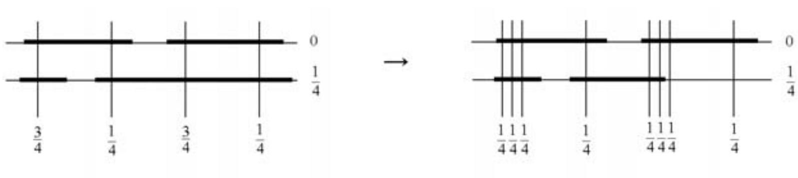

FIG. 3. *Example of an initial instance (left) and a new instance satisfying the assumption (right).*

We order the rows of the grid in order of nonincreasing z^{lp}-values, and we denote by l ($l \geq 0$) the number of z^{lp}-values equal to 1. Then $z^{\mathrm{lp}}_1 = \cdots = z^{\mathrm{lp}}_l = 1, 1 > z^{\mathrm{lp}}_{l+1} \geq \cdots \geq z^{\mathrm{lp}}_m \geq 0$. We assume that $value(y^{\mathrm{lp}})$ is positive (otherwise all the z^{lp}-values have to be equal to 1 and the theorem obviously holds).

By construction,

$$\text{(18)} \qquad value(y^*, z^*) = \min_{j \in [0:m]} value(y^j, z^j) \leq \min_{j \in [l:m]} value(y^j, z^j),$$

where (y^j, z^j) is the jth solution generated in step 4 of *STAB*.

Let us proceed by defining a number $q_j = q_j(\Delta, \beta) \in \mathbb{R}$ for each $j \in [0 : m]$ that depends on a given $\Delta \in [0, 1]^m$ and $\beta > 0$ as follows:

$$\sum_{k=1}^{\lfloor q_j \rfloor} (1 - \Delta_{j+k}) + (q_j - \lfloor q_j \rfloor)(1 - \Delta_{j+\lceil q_j \rceil}) = \beta, \tag{19}$$

where we put $\Delta_j = 0$ if $j > m$. Since the left-hand side is 0 at $q_j = 0$ and continuously increases to infinity as q_j grows, there always exists a unique point q_j satisfying the equality.

We will prove the following lemma.

LEMMA 3.2.

$$value(y^j, z^j) \leq j + k \cdot q_j \left(z^{lp}, \frac{value(y^{lp})}{k} \right) \quad \forall j \in [l : m].$$

Then, assuming that Lemma 3.2 holds, it follows from (18) that

$$value(y^*, z^*) \leq \min_{j \in [l:m]} \left(j + k \cdot q_j \left(z^{\text{lp}}, \frac{value(y^{\text{lp}})}{k} \right) \right). \tag{20}$$

Theorem 3.1 follows now from the following lemma, the proof of which can be found in the appendix.

LEMMA 3.3. *Given are real numbers* $1 \geq \Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_m \geq 0$, *a positive real number* Y, *an integer* $p \geq 2$, *and an integer* $l \geq 0$. *Then the following holds:*

$$\min_{i \in [l:m]} (i + p \cdot q_i(\Delta, Y/p)) \leq \frac{1}{1 - (1 - 1/p)^p} \left(Y + \sum_{r=l+1}^{m} \Delta_r \right) + l. \tag{21}$$

By applying this lemma with $p = k$, $\Delta = z^{\text{lp}}$, and $Y = value(y^{\text{lp}})$, the right-hand side of (20) can be bounded by

$$\frac{1}{1-(1-1/k)^k} \left(value(y^{\text{lp}}) + \sum_{r=l+1}^{m} z_r^{\text{lp}} \right) + l \leq \frac{1}{1-(1-1/k)^k} \left(value(y^{\text{lp}}) + \sum_{r=l+1}^{m} z_r^{\text{lp}} + l \right),$$

and since $z_1^{\text{lp}} = \cdots = z_l^{\text{lp}} = 1$, the right-hand side of this last expression is equal to

$$\frac{1}{1 - (1 - 1/k)^k} value(y^{\text{lp}}, z^{\text{lp}}).$$

The theorem is then proved.

Proof of Lemma 3.2. Consider (y^j, z^j); for some $j \in [l : m]$, let us find an upper bound for $value(y^j, z^j)$. By construction,

– $z_r^j = 1 \ \forall r \leq j$,

– $z_r^j = 0 \ \forall r \geq j + 1$,

– y^j is an optimal solution to $\text{IP}^y(\mathcal{I}, b)$, where $b_i = 1 - z_{\rho_i}^j \ \forall i \in [1 : n]$.

Obviously, $value(z^j) = j$. In order to bound $value(y^j)$ we introduce a solution y'^j, which is feasible to the LP-relaxation of $\text{IP}^y(\mathcal{I}, b)$. Then, Corollary 2.2 implies that $value(y^j) \leq value(y'^j)$.

First, let us define subsets $S_1, S_2, \ldots, S_m$, where $S_r \subset [1:t]$ $\forall r = 1, \ldots, m$ (i.e., each subset consists of a set of columns of the grid), in the following way:

$$S_r = \bigcup_{i:\rho_i = r} [l_i : r_i].$$

Thus, S_r is the set of columns stabbing intervals in row r.

Fix now some $j \in [l : m]$, and construct vector y'^j as follows: For each column $c \in [1:t]$,

– if $c \in S_{j+1} \cup \cdots \cup S_m$, then denote by t the minimum index such that $c \in S_t$ and let $y'^j_c = \frac{1}{(1-z^{\text{lp}}_t)} y^{\text{lp}}_c$ (recall that $z^{\text{lp}}_r < 1$ $\forall r \in [l+1 : m]$);

– otherwise, let $y'^j_c = y^{\text{lp}}_c$.

Let us now establish feasibility of y'^j with respect to the LP-relaxation of $\text{IP}^y(\mathcal{I}, b)$. For any interval i we show that the following inequality holds:

$$\sum_{c \in [l_i : r_i]} y'^j_c \geq 1 - z^j_{\rho_i}. \tag{22}$$

If $\rho_i < j + 1$, where ρ_i is the row number of interval i, then $z^j_{\rho_i} = 1$, and the inequality holds automatically. Consider the case $\rho_i \geq j + 1$. For any $c \in S_{\rho_i}$, y'^j_c is defined as $y^{\text{lp}}_c / (1 - z^{\text{lp}}_t)$, where $t \leq \rho_i$. Since z^{lp}_t are nonincreasing with t, we have $y'^j_c \geq y^{\text{lp}}_c / (1 - z^{\text{lp}}_{\rho_i})$. Then, since $[l_i : r_i] \subseteq S_{\rho_i}$, we have $y'^j_c \geq y^{\text{lp}}_c / (1 - z^{\text{lp}}_{\rho_i})$ for any $c \in [l_i : r_i]$. Using this, and remembering that $(y^{\text{lp}}, z^{\text{lp}})$ satisfies $z^{\text{lp}}_{\rho_i} + \sum_{c \in [l_i : r_i]} y^{\text{lp}}_c \geq 1$, we have

$$\sum_{c \in [l_i : r_i]} y'^j_c \quad \geq \quad \frac{1}{(1 - z^{\text{lp}}_{\rho_i})} \sum_{c \in [l_i : r_i]} y^{\text{lp}}_c \quad \geq \quad \frac{1 - z^{\text{lp}}_{\rho_i}}{1 - z^{\text{lp}}_{\rho_i}} = 1.$$

Thus, we have shown that inequality (22) holds for any $i \in [1 : n]$, and therefore y'^j is feasible to the LP-relaxation of $\text{IP}^y(\mathcal{I}, b)$. Now Corollary 2.2 implies that

$$value(y^j) \leq value(y'^j). \tag{23}$$

In what follows we show that $value(y'^j) \leq k \cdot q_j(z^{\text{lp}}, \frac{value(y^{\text{lp}})}{k})$ $\forall j \in [l : m]$. By construction of y'^j, using notation $Y(S) = \sum_{c \in S} y^{\text{lp}}_c$,

(24)
$$value(y'^j) = \tfrac{1}{1 - z^{\text{lp}}_{j+1}} Y(S_{j+1}) \; + \; \tfrac{1}{1 - z^{\text{lp}}_{j+2}} Y(S_{j+2} \backslash S_{j+1})$$
$$+ \cdots + \tfrac{1}{1 - z^{\text{lp}}_m} Y(S_m \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_{m-1})) + Y([1:t] \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_m)).$$

Observe that for the $Y(\cdot)$-terms the following equality holds:

$$Y(S_{j+1}) + Y(S_{j+2} \backslash S_{j+1}) + \cdots + Y(S_m \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_{m-1}))$$
$$+ Y([1:t] \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_m)) = \sum_{c=1}^{t} y^{\text{lp}}_c = value(y^{\text{lp}}). \tag{25}$$

Moreover, using the definition of S_r, our assumption (17), and the fact that there are at most k intervals per row, we have for each $r = j+1, \dots, m$

$$Y(S_r \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_{r-1})) \le Y(S_r) = \sum_{c \in S_r} y_c^{\text{lp}}$$

$$\le \sum_{i:\rho_i = r} \sum_{c \in [l_i : r_i]} y_c^{\text{lp}} = \sum_{i:\rho_i = r} (1 - z_{\rho_i}^{\text{lp}}) \le k(1 - z_r^{\text{lp}}). \tag{26}$$

Now consider the following optimization problem:

$$\max_{Y_{j+1}, Y_{j+2}, \dots} \left(\frac{1}{1 - z_{j+1}^{\text{lp}}} Y_{j+1} + \frac{1}{1 - z_{j+2}^{\text{lp}}} Y_{j+2} + \cdots + \frac{1}{1 - z_m^{\text{lp}}} Y_m + \sum_{r=m+1}^{\infty} Y_r \right)$$

$$\text{subject to} \quad Y_{j+1} + \cdots + Y_m + \sum_{r=m+1}^{\infty} Y_r \le value(y^{\text{lp}}), \tag{27}$$

$$0 \le Y_r \le k(1 - z_r^{\text{lp}}) \qquad \forall r = j+1, \dots, m, \tag{28}$$

$$0 \le Y_r \le k \qquad \forall r = m+1, \dots, \infty. \tag{29}$$

Due to (25) and (26) the following solution is feasible to this optimization problem: $Y_r = Y(S_r \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_{r-1}))$ for each $r = j+1, \dots, m$, and $\sum_{r=m+1}^{\infty} Y_r = Y((1:t) \backslash (S_{j+1} \cup S_{j+2} \cup \cdots \cup S_m))$ (distributed arbitrarily among the components of the sum while satisfying (29)). Therefore the optimum value of this optimization problem is an upper bound on the right-hand side of (24).

What does the optimum solution to this optimization problem look like? Notice that the constraint matrix of (27)–(29) is a so-called *greedy* matrix (see Hoffman, Kolen, and Sakarovitch [5]). Together with the fact that the objective coefficients are nonincreasing, a result from [5] implies that successive maximization of the variables $Y_{j+1}, Y_{j+2}, \dots$ in this order produces an optimum solution. Thus, we obtain the following optimal solution:

$$Y_{j+1} = k(1 - z_{j+1}^{\text{lp}}), \; Y_{j+2} = k(1 - z_{j+2}^{\text{lp}}), \dots, \quad Y_{j+\lfloor q \rfloor} = k(1 - z_{j+\lfloor q \rfloor}^{\text{lp}}),$$

$$Y_{j+\lfloor q \rfloor + 1} = (q - \lfloor q \rfloor) k (1 - z_{j+\lfloor q \rfloor + 1}^{\text{lp}})$$

for some number $q \in \mathbb{R}_+$, which due to (27) has to satisfy

$$k(1 - z_{j+1}^{\text{lp}}) + k(1 - z_{j+2}^{\text{lp}}) + \cdots + k(1 - z_{j+\lfloor q \rfloor}^{\text{lp}}) + k(q - \lfloor q \rfloor)(1 - z_{j+\lfloor q \rfloor + 1}^{\text{lp}}) = value(y^{\text{lp}}),$$

where we put $z_r^{\text{lp}} = 0$ for any $r > m$. Notice that $q \equiv q_j(z^{\text{lp}}, \frac{value(y^{\text{lp}})}{k})$ (see (19)), and the optimum value of the problem (27)–(29), which bounds the right-hand side of (24) from above, is $k \cdot q_j(z^{\text{lp}}, \frac{value(y^{\text{lp}})}{k})$. This proves Lemma 3.2. ∎

3.2. The approximation result for WISP. In this section we consider the weighted version of ISP, without any limitation on the number of rectangles sharing a row, and prove the following result.

THEOREM 3.4. *Algorithm STAB is an* $e/(e-1) \approx 1.582$*-approximation algorithm for WISP.*

Proof. Consider an instance $\mathcal{I}$ of WISP, and let $(y^{\text{lp}}, z^{\text{lp}})$ and (y^*, z^*) be, respectively, an optimal solution to the LP-relaxation of (10)–(12) and the solution returned by the algorithm for $\mathcal{I}$. We show that their values are related as follows:

$$value(y^*, z^*) \le \frac{e}{e-1} value(y^{\text{lp}}, z^{\text{lp}}). \tag{30}$$

Since $value(y^{\text{lp}}, z^{\text{lp}})$ is a lower bound for the optimal value of WIS, the theorem follows.

Assume, without loss of generality, that the rows of the grid are sorted so that $z_1^{\text{lp}} \geq z_2^{\text{lp}} \geq \cdots \geq z_m^{\text{lp}}$. Further, suppose there are l z^{lp}-values equal to 1, i.e., $z_1^{\text{lp}} = \ldots z_l^{\text{lp}} = 1$, and $1 > z_{l+1}^{\text{lp}} \geq z_{p+2}^{\text{lp}} \geq \cdots \geq z_m^{\text{lp}} \geq 0$.

Let (y^j, z^j) be candidate solution number j constructed by *STAB* for $\mathcal{I}$ $\forall j \in [0 : m]$. From the design of *STAB* we know that

$$value(y^*, z^*) = \min_{j \in [0:m]} value(y^j, z^j) \leq \min_{j \in [l:m]} value(y^j, z^j). \tag{31}$$

Claim 1.

$$value(y^j, z^j) \equiv wy^j + vz^j \leq \sum_{r=1}^{j} v_r + \frac{wy^{\text{lp}}}{1 - z_{j+1}^{\text{lp}}} \quad \text{for any } j \in [l : m].$$

Let us prove it. Consider (y^j, z^j) for some $j \in [l : m]$. By construction,

– $z_r^j = 1$ $\forall r \leq j$,

– $z_r^j = 0$ $\forall r \geq j + 1$,

– y^j is an optimal solution to $\text{IP}^y(\mathcal{I}, b)$ with $b_i = 1 - z_{\rho_i}$ $\forall i \in [1 : n]$.

Clearly, $vz^j \equiv \sum_{r=1}^{m} v_r z_r^j = \sum_{r=1}^{j} v_r$. Let us show that

$$wy^j \leq \frac{wy^{\text{lp}}}{1 - z_{j+1}^{\text{lp}}}. \tag{32}$$

To prove this, we establish that the fractional solution

$$\frac{1}{1 - z_{j+1}^{\text{lp}}}\, y^{\text{lp}}, \tag{33}$$

where we set $z_{m+1}^{\text{lp}} = 0$, is feasible to the LP-relaxation of $\text{IP}^y(\mathcal{I}, b)$. Since y^j is optimal to $\text{IP}^y(\mathcal{I}, b)$, Corollary 2.2 implies (32). So, let us prove the following claim.

Claim 1.1. Solution (33) is feasible to the LP-relaxation of $\text{IP}^y(\mathcal{I}, b)$ with $b_i = 1 - z_{\rho_i}$ $\forall i \in [1 : n]$. We show that constraint (8) is satisfied:

$$\frac{1}{1 - z_{j+1}^{\text{lp}}} \sum_{c \in [l_i, r_i]} y_c^{\text{lp}} \geq 1 - z_{\rho_i}^j \quad \text{for any } i \in [1 : n]. \tag{34}$$

Indeed, in case $z_{\rho_i}^j = 1$, the inequality trivially holds. Otherwise, if $z_{\rho_i}^j = 0$, it follows from the construction of z^j that $\rho_i \geq j+1$. The ordering of the z^{lp}-values implies that $z_{\rho_i}^{\text{lp}} \leq z_{j+1}^{\text{lp}}$. Then, using this and the fact that solution $(y^{\text{lp}}, z^{\text{lp}})$ satisfies constraint (14), we have

$$\frac{1}{1 - z_{j+1}^{\text{lp}}} \sum_{c \in [l_i, r_i]} y_c^{\text{lp}} \geq \frac{1}{1 - z_{j+1}^{\text{lp}}}(1 - z_{\rho_i}^{\text{lp}}) \geq \frac{1}{1 - z_{j+1}^{\text{lp}}}(1 - z_{j+1}^{\text{lp}}) = 1.$$

This proves (34) and, subsequently, Claims 1.1 and 1.

From (31) and Claim 1,

$$value(y^*, z^*) \leq \min_{j \in [l:m]} \left(\sum_{r=1}^{j} v_r + \frac{wy^{\text{lp}}}{1 - z_{j+1}^{\text{lp}}} \right)$$

$$= \sum_{r=1}^{l} v_r + \min_{j\in[l:m]} \left(\sum_{r=l+1}^{j} v_r + \frac{wy^{\text{lp}}}{1 - z_{j+1}^{\text{lp}}} \right).$$

Lemma 3.5 given below implies now that the last expression can be upper bounded by

$$\sum_{r=1}^{l} v_r + \frac{e}{e-1} \left(\sum_{r=l+1}^{m} v_r z_r^{\text{lp}} + wy^{\text{lp}} \right) \leq \frac{e}{e-1} \left(\sum_{r=1}^{l} v_r + \sum_{r=l+1}^{m} v_r z_r^{\text{lp}} + wy^{\text{lp}} \right).$$

Since $z_1^{\text{lp}} = \cdots = z_l^{\text{lp}} = 1$, the last expression can be rewritten as

$$\frac{e}{e-1} \left(\sum_{r=1}^{m} v_r z_r^{\text{lp}} + wy^{\text{lp}} \right) = \frac{e}{e-1} (vz^{\text{lp}} + wy^{\text{lp}}),$$

which establishes inequality (30) and proves the theorem. ∎

LEMMA 3.5. *Suppose we are given numbers* $1 > \Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_m \geq 0$ $\forall i = 1, \ldots, m$, *and* $\Delta_{m+1} = 0$. *Further, given are positive numbers* $a_1, a_2, \ldots, a_m$ *and* Y. *Then we have*

$$\min_{j=0,\ldots,m} \left(\sum_{r=1}^{j} a_r + \frac{1}{1 - \Delta_{j+1}} Y \right) \leq \frac{e}{e-1} \left(\sum_{r=1}^{m} a_r \Delta_r + Y \right). \tag{35}$$

We give the proof of this lemma in the appendix.

3.3. Tightness. In this subsection we demonstrate that the ratio between the optimum values of ISP_k and the LP-relaxation of its ILP formulation (13)–(15) can be arbitrarily close to the bounds achieved by *STAB* in case $k = 2$ and $k = \infty$ (which are, respectively, $4/3$ and $e/(e-1)$).

For the case $k = 2$ this is shown by the instance of ISP_2 depicted in Figure 4 (recall that all the column and row demands and rectangle weights are unit). Here the optimal value of the problem is 2, since at least two elements (columns or rows) are needed to stab the three rectangles, whereas the optimal fractional solution has the value of 3/2.

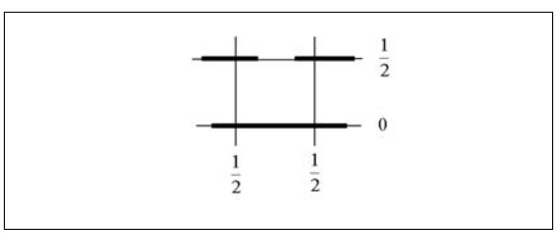

FIG. 4. *An instance of ISP_2 and an optimal fractional solution.*

In the remainder of the section we consider the problem ISP_∞, or simply ISP, without any limitation on the number of rectangles sharing a row.

THEOREM 3.6. *The integrality gap of* (13)–(15) *is arbitrarily close to* $\frac{e}{e-1}$.

Proof. For each $m \in \mathbb{N}$ we will construct an instance $\mathcal{I}_m$ of ISP and show that the value of some feasible solution to its LP-relaxation tends to be $\frac{e}{e-1}$ times its optimal value as m increases.

Let us construct $\mathcal{I}_m$ as follows. Let the grid have m rows and $t = m!$ columns. Let the rows be numbered consecutively and let each row j intersect exactly j rectangles of the instance. Let rectangles intersected by row j be numbered $j_1, \ldots, j_j$. All these rectangles are disjoint and each intersects exactly $\frac{m!}{j}$ columns (see Figure 5). So, for a rectangle j_i we have that its row number ρ_{j_i} is r, and its leftmost and rightmost columns are $l_{j_i} = \frac{m!}{j}(i-1)+1$ and $r_{j_i} = \frac{m!}{j}i$. The total number of rectangles in the instance is then $n = 1 + 2 + \cdots + m$.

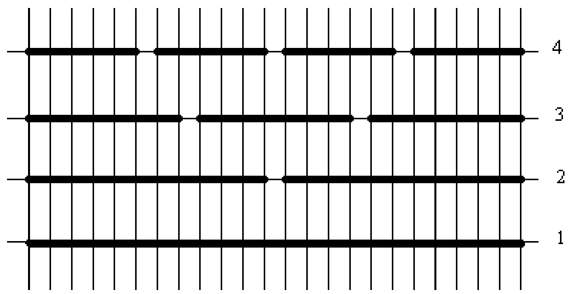

FIG. 5. *Instance* $\mathcal{I}_4$.

We claim that the following solution (y, z) is feasible to the LP-relaxation of (13)–(15) for $\mathcal{I}_m$:

$$(36) \qquad \begin{aligned} z_j &= \begin{cases} 0 & \forall j = 1, \ldots, P, \\ 1 - P/j & \forall j = P+1, \ldots, m, \end{cases} \\ y_c &= \frac{P}{m!} \quad \forall c = 1, \ldots, m!, \end{aligned}$$

where $P = P(m)$ is the number satisfying

$$\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P+1} \leq 1 \ \text{ and } \ \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P+1} + \frac{1}{P} \geq 1.$$

Denote the value of this solution by $LP(\mathcal{I}_m)$, and observe that

$$LP(\mathcal{I}_m) = \sum_{c=1}^{t} y_c + \sum_{r=1}^{m} z_r = m - P\left(\frac{1}{P+1} + \frac{1}{P+2} + \cdots + \frac{1}{m}\right).$$

Let us show feasibility of (y, z). Take any rectangle j_i and show that the constraint $z_{\rho_{j_i}} + \sum_{c \in [l_{j_i}, r_{j_i}]} y_c \geq 1$ is satisfied. Notice that the z-values of our solution also can be expressed as $z_j = \max(1 - \frac{P}{j}, 0)\ \forall j = 1, \ldots, m$. Substituting these values, and rewriting the left-hand side of constraints (14) gives

$$\begin{aligned} \max\left(1 - \frac{P}{j_i}, 0\right) + \sum_{c \in [l_{j_i}, r_{j_i}]} \frac{P}{m!} &= \max\left(1 - \frac{P}{j_i}, 0\right) + \frac{m!}{j_i}\frac{P}{m!} \\ &= \max\left(1 - \frac{P}{j_i}, 0\right) + \frac{P}{j_i}. \end{aligned}$$

Clearly, the last expression is at least equal to 1, which proves feasibility of solution (y, z) to the LP-relaxation of (13)–(15) for $\mathcal{I}_m$.

Now denote by $OPT(\mathcal{I})$ the optimum value to ISP for $\mathcal{I}$, and show that $OPT(\mathcal{I}_m) = m$. Consider any optimal integral solution, and denote by k the maximum row number, whose corresponding z-value is 0. First, this means that there are at least $m-k$ rows whose z-values are 1. Second, observe that, since there are k disjoint rectangles on row k and this row is not selected, there are at least k columns needed to stab these rectangles. Therefore, this solutions has to select at least $m-k$ rows and k columns, meaning $OPT(\mathcal{I}_m) \geq m$. Since there exists a feasible solution of value m (select all the rows, for instance), we obtain that $OPT(\mathcal{I}_m) = m$.

We use Lemma 5.3 given in the appendix to prove that the ratio

$$\frac{OPT(\mathcal{I}_m)}{LP(\mathcal{I}_m)} = \frac{m}{m - P(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P+1})}$$

approaches $\frac{e}{e-1}$ when m increases. This establishes our tightness result. ☐

As mentioned in the introduction, Theorems 3.1 and 3.6 imply that it is unlikely that a better ratio for ISP_∞ can be achieved using formulation (13)–(15).

Approximation algorithms with a ratio of $\frac{e}{e-1}$ are not uncommon in the literature; integrality gaps with this ratio seem to appear less frequently. Another example of a (different) formulation with an integrality gap that equals $\frac{e}{e-1}$ is described in Hoogeveen, Skutella, and Woeginger [6].

4. Conclusion. We presented an approximation algorithm called *STAB* for two variants of the weighted rectangle stabbing problem. *STAB* achieves a ratio of $\frac{1}{1-(1-1/k)^k}$ for ISP_k, the special case where each rectangle is stabbed by a single row and by at most k columns, and where all stabbing lines have unit weight. *STAB* achieves a ratio of $\frac{e}{e-1}$ for WISP, the special case where each rectangle is stabbed by a single row. *STAB* considers different ways of rounding the LP-relaxation and outputs the best solution found in this way; it is also shown that the ratio proved equals the integrality gap when $k = 2$ and when $k = \infty$.

5. Appendix. In this appendix we give proofs of lemmas which we used in this paper.

LEMMA 3.3. *Given are real numbers* $1 \geq \Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_m \geq 0$, *a positive real number* Y, *an integer* $p \geq 2$, *and an integer* $0 \leq l < m$. *The following holds:*

$$\min_{i=l,\dots,m} (i + p \cdot q_i(\Delta, Y/p)) \leq \frac{1}{1-(1-1/p)^p} \left(Y + \sum_{r=l+1}^{m} \Delta_r \right) + l, \tag{37}$$

where $q_i = q_i(\Delta, Y/p)$ *for each* $i \in [0:m]$ *is uniquely defined by the equality*

$$\sum_{k=1}^{\lfloor q_i \rfloor} (1 - \Delta_{i+k}) + (q_i - \lfloor q_i \rfloor)(1 - \Delta_{i+\lceil q_i \rceil}) = Y/p, \tag{38}$$

where we put $\Delta_i = 0$ *if* $i > m$.

Proof. It is enough to prove this lemma for $l = 0$. The case of other $l < m$ can be reduced to the case of $l = 0$ by changing the index to $j = i - l$ and observing that $q_{j+l}(\Delta, Y/p) = q_i(\Delta^{-l}, Y/p)$, where vector Δ^{-l} is obtained by deleting the first l elements from vector Δ. So we will prove that

$$\min_{i=0,\dots,m} (i + p \cdot q_i(\Delta, Y/p)) \leq \frac{1}{1-(1-1/p)^p} \left(Y + \sum_{r=1}^{m} \Delta_r \right).$$

The proof consists of two lemmas. In Lemma 5.1 we show that the left-hand side of (37) is upper bounded by the following supremum:

$$\sup_{f(\cdot) \in H} G(f(\cdot)), \tag{39}$$

where

$$G(f(\cdot)) = \min_{x \in \mathbb{R}_+} \left(f(x) + p \cdot (f(x + Y/p) - f(x))\right), \tag{40}$$

and the class of functions H is defined as

$$H = \left\{ f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \;\middle|\; \begin{array}{l} f(\cdot) \text{ is continuous, increasing, concave,} \\ f(0) = 0, f(x) \leq x + \sum_{r=1}^{m} \Delta_r \end{array} \right\}. \tag{41}$$

In Lemma 5.2 we show that this supremum is upper bounded by the right-hand side of (37), which proves the lemma.

LEMMA 5.1.

$$\min_{i=0,\dots,m} (i + p \cdot q_i(\Delta, Y/p)) \leq \sup_{f(\cdot) \in H} G(f(\cdot)),$$

where $G(f(\cdot))$ *and* H *are defined in* (40) *and* (41).

Proof. To establish this, it is sufficient to exhibit a particular function $\hat{f}(\cdot) \in H$, such that

$$G(\hat{f}(\cdot)) = \min_{i=0,\dots,m} (i + p \cdot q_i(\Delta, Y/p)). \tag{42}$$

Then, the supremum of $G(f(\cdot))$ over all the possible $f(\cdot) \in H$ is clearly larger than or equal to $G(\hat{f}(\cdot))$.

Before we describe the function $\hat{f}(\cdot)$, let us define an auxiliary function $F(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$F(q) \equiv \sum_{r=1}^{\lfloor q \rfloor} (1 - \Delta_r) + (q - \lfloor q \rfloor)\ (1 - \Delta_{\lceil q \rceil}), \tag{43}$$

where we set $\Delta_r = 0\ \forall r \geq m + 1$.

Observe that $F(\cdot)$ is

– continuous;

– increasing, since $\Delta_r < 1$, and therefore $(1 - \Delta_r) > 0\ \forall r = 1, \dots, \infty$;

– convex, since the coefficients Δ_r are nonincreasing with increasing r, and therefore the coefficients $(1 - \Delta_r)$ are nondecreasing with increasing r.

Furthermore,

– $F(0) = 0$;

– $F(q) \geq (q - \sum_{r=1}^{m} \Delta_r)\ \forall q \in \mathbb{R}_+$, since $F(q)$ can be also represented as

$$F(q) = q - \left(\sum_{r=1}^{\lfloor q \rfloor} \Delta_r + (q - \lfloor q \rfloor)\Delta_{\lceil q \rceil} \right),$$

and obviously $(\sum_{r=1}^{\lfloor q \rfloor} \Delta_r + (q - \lfloor q \rfloor)\Delta_{\lceil q \rceil}) \leq \sum_{r=1}^{m} \Delta_r\ \forall q \in \mathbb{R}_+$;

– $F(q)$ is linear on each of the intervals $[i, i+1]$, $i = 0, \dots, m-1$, and on $[m, +\infty)$.

We are now ready to present $\hat{f}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$. We define

$$\hat{f}(\cdot) \equiv F^{-1}(\cdot)$$

(since $F(\cdot)$ is increasing, $F^{-1}(\cdot)$ exists).

We claim that $\hat{f}(\cdot) \in H$. Indeed, $\hat{f}(\cdot)$ has the following properties:

– $\hat{f}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ since $F(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$;

– $\hat{f}(\cdot)$ is continuous, increasing, and concave, since $F(\cdot)$ is continuous, increasing, and convex;

– $\hat{f}(0) = 0$, since $F(0) = 0$;

– $\hat{f}(x) \leq x + \sum_{r=1}^{m} \Delta_r \ \forall x \in \mathbb{R}_+$. This can be obtained from $F(q) \geq (q - \sum_{r=1}^{m} \Delta_r) \ \forall q \in \mathbb{R}_+$, using $F(q) = x$, $q = \hat{f}(x)$.

This proves that $\hat{f}(\cdot) \in H$.

To prove the lemma it remains to show that

$$G(\hat{f}(\cdot)) = \min_{i=0,\dots,m} (i + p \cdot q_i(\Delta, Y/p)).$$

Comparing the definition of $q_i(\Delta, Y/p)$ (see (38)) and $F(\cdot)$ (see (43)), observe that for each $i \in [0 : m]$ q_i satisfies

$$F(i + q_i) - F(i) = Y/p. \tag{44}$$

Thus, $q_i = F^{-1}(F(i) + Y/p) - i$. Setting $x_i \equiv F(i)$ $\forall i = 0, \dots, m$, we find that $i = F^{-1}(x_i)$ and $q_i = F^{-1}(x_i + Y/p)) - F^{-1}(x_i)$. Replacing $F^{-1}(\cdot)$ by $\hat{f}(\cdot)$, we obtain

$$q_i = \hat{f}(x_i + Y/p) - \hat{f}(x_i) \ \forall i = 0, \dots, m.$$

Using this together with $i = F^{-1}(x_i) = \hat{f}(x_i)$, we can rewrite

$$\min_{i=0,\dots,m} (i + p \cdot q_i(\Delta, Y/p)) = \min_{\substack{i = 0, \dots, m \\ x_i = \hat{f}^{-1}(i)}} \Big(\hat{f}(x_i) + p(\hat{f}(x_i + Y/p) - \hat{f}(x_i)) \Big). \tag{45}$$

Now we need to show that the latter expression is equal to

$$G(\hat{f}(\cdot)) \equiv \min_{x \in \mathbb{R}_+} \Big(\hat{f}(x) + p(\hat{f}(x + Y/p) - \hat{f}(x)) \Big). \tag{46}$$

We do this by showing that the function $\hat{f}(x) + p(\hat{f}(x + Y/p) - \hat{f}(x))$ is continuous and concave in each of the intervals $[x_i, x_{i+1}]$ $\forall i = 0, \dots, m-1$, and is increasing in $[x_m, +\infty)$. Therefore the minimum can be achieved only at one of the endpoints $x_0, x_1, \dots, x_m$.

Indeed, consider function $\hat{f}(x) + p(\hat{f}(x+Y/p) - \hat{f}(x))$ in $[x_i, x_{i+1}]$ for some $i \in [0 : m-1]$. It can also be written as $p\hat{f}(x + Y/p) - (p-1)\hat{f}(x)$. We know that $\hat{f}(x + Y/p)$ is concave on $[x_i, x_{i+1}]$, since it is concave everywhere in $\mathbb{R}_+$. Furthermore, $\hat{f}(x)$ is linear on each $[x_i, x_{i+1}], i \in [0 : m-1]$, since $F(\cdot)$ is linear on $[i, i+1]$, $i \in [0 : m-1]$. Obviously, a concave function minus a linear function is again concave.

Now we show that $p\hat{f}(x + Y/p) - (p-1)\hat{f}(x)$ is increasing in $[x_m, +\infty)$. Since $\hat{f}(x) = F^{-1}(\cdot)$ is increasing and linear in $[x_m, +\infty)$, the growth rate of $\hat{f}(x)$ is the same as the growth rate of $\hat{f}(x + Y/p)$ in $[x_m, +\infty)$, and thus the growth rate of $p\hat{f}(x + Y/p) - (p - 1)\hat{f}(x)$ is positive. We have proved that the minimum in (46) is always achieved at one of the points $x_0, x_1, \ldots, x_m$, and therefore (46) is equal to (45). This completes the proof of Lemma 5.1. ∎

Lemma 5.2.

$$\sup_{f(\cdot)\in H} G(f(\cdot)) \le \frac{1}{1 - (1 - 1/p)^p} C,$$

where

$$C = Y + \sum_{r=1}^{m} \Delta_r,$$

$$G(f(\cdot)) = \min_{x\in\mathbb{R}_+} \left(f(x) + p(f(x+Y/p) - f(x))\right),$$

and the set of functions H (via notation C) is

$$H = \left\{ f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \;\middle|\; \begin{array}{l} f(\cdot) \textit{ is continuous, increasing, concave,} \\ f(0) = 0, f(x) \le x + C - Y \end{array} \right\}.$$

Proof. We will prove several claims and subclaims.

Claim 1.

$$\sup_{f(\cdot)\in H} G(f(\cdot)) = \sup_{g\,:\,f^g(\cdot)\in H} g,$$

where for each $g \in \mathbb{R}_+$ function $f^g(\cdot)$ is defined as follows:

– $f^g(j \cdot Y/p) = g(1 - (1 - 1/p)^j)$ $\forall j \in 0 \cup \mathbb{N}$;

– $f^g(x)$ is continuous in $[0, +\infty)$ and linear in each $[(j - 1) \cdot Y/p, j \cdot Y/p]$, $j \in \mathbb{N}$.

Notice that $f^g(\cdot)$ is completely defined by the above characterization.

To prove this claim it is enough to show that for any $f(\cdot) \in H$ there exists a function $f^{\hat{g}}(\cdot) \in H$, with $\hat{g} \ge 0$, such that

$$G(f(\cdot)) = G(f^{\hat{g}}(\cdot)) = \hat{g}.$$

To show that, we prove two subsidiary claims.

Claim 1.1. For any $g \ge 0$,

$$G(f^g(\cdot)) \equiv \min_{x\in\mathbb{R}_+} \left(f^g(x) + p(f^g(x+Y/p) - f^g(x))\right) = g.$$

Indeed, by construction $f^g(x)$ is linear in each of the intervals $[(j - 1) \cdot Y/p, j \cdot Y/p]$, $j \in \mathbb{N}$. This implies that function $(f^g(x) + p(f^g(x+Y/p) - f^g(x)))$ is linear in each of these intervals as well. Therefore the minimum over all $x \ge 0$ is achieved in one of the endpoints 0, Y/p, $2Y/p, \ldots$. Consider $(f^g(x) + p \cdot (f^g(x+Y/p) - f^g(x)))$ at the point $x = j \cdot Y/p$ for some $j \in \mathbb{N} \cup 0$:

$$f^g(j \cdot Y/p) + p \cdot (f^g((j + 1) \cdot Y/p) - f^g(j \cdot Y/p)).$$

Using the definition of $f^g(\cdot)$ we can rewrite it as follows:

$$g \cdot (1-(1-1/p)^j) + p \cdot (g(1-(1-1/p)^{j+1}) - g \cdot (1-(1-1/p)^j)).$$

With simple computations one can verify that the last expression is equal to g. This proves Claim 1.1.

Claim 1.2. For any $f(\cdot) \in H$ it holds that $f^{\hat{g}}(\cdot) \in H$, where $\hat{g} = G(f(\cdot))$.

Clearly, $f^{\hat{g}}(x)$ is concave. To prove that $f^{\hat{g}}(x) \leq x + C - Y \ \forall x \in \mathbb{R}_+$, it is sufficient to show that $f^{\hat{g}}(x) \leq f(x)$, since $f(\cdot) \in H$ means, e.g., $f(x) \leq x + C - Y \ \forall x \in \mathbb{R}_+$.

So, let us establish that $f^{\hat{g}}(x) \leq f(x) \ \forall x \in \mathbb{R}_+$. Recall that $f^{\hat{g}}(x)$ is linear in each of the intervals $[(j-1) \cdot Y/p, j \cdot Y/p]$, $j \in \mathbb{N}$, and $f(x)$ is concave in $\mathbb{R}_+$. Then it is sufficient to show that

$$f^{\hat{g}}(x) \leq f(x) \ \forall x = j \cdot Y/p, \ j \in 0 \cup \mathbb{N}.$$

We use mathematical induction on j. For $j = 0$, $f^{\hat{g}}(0) = f(0) = 0$ and the inequality trivially holds. Suppose, for $j-1$ we have proved that $f^{\hat{g}}((j-1) \cdot Y/p) \leq f((j-1) \cdot Y/p)$, and let us show that $f^{\hat{g}}(j \cdot Y/p) \leq f(j \cdot Y/p)$.

Observe that $f^{\hat{g}}(\cdot)$ can be represented in a recursive way as follows:

$$f^{\hat{g}}(j \cdot Y/p) = \hat{g}/p + f^{\hat{g}}((j-1) \cdot Y/p) \ (1-1/p). \tag{47}$$

Since $\hat{g} = G(f(\cdot))$ we know that

$$\hat{g} \leq f((j-1) \cdot Y/p) + p \cdot (f(j \cdot Y/p) - f((j-1) \cdot Y/p)).$$

Rearranging the expression, we obtain

$$f(j \cdot Y/p) \geq \hat{g}/p + f((j-1) \cdot Y/p) \ (1-1/p).$$

By the induction hypothesis and (47) we can bound the right-hand side by

$$\hat{g}/p + f((j-1) \cdot Y/p) \ (1-1/p) \ \geq \ \hat{g}/p + f^{\hat{g}}((j-1) \cdot Y/p) \ (1-1/p) = f^{\hat{g}}(j \cdot Y/p).$$

This proves Claim 1.2.

These two claims imply that for any $f(\cdot) \in H$, there exists $f^{\hat{g}}(\cdot) \in H$, with $\hat{g} \geq 0$, such that

$$G(f(\cdot)) = G(f^{\hat{g}}(\cdot)) = \hat{g}.$$

This implies Claim 1.

Claim 2.

$$\sup_{g \,:\, f^g(\cdot) \in H} g \leq \frac{1}{1-(1-1/p)^p} \ C.$$

Indeed, $f^g(\cdot) \in H$ implies $f^g(x) \leq x + C - Y \ \forall x \in \mathbb{R}_+$ and, in particular, for $x = Y$. From this, using the definition of $f^g(\cdot)$, we obtain

$$f^g(Y) \equiv f^g(p \cdot Y/p) \equiv g(1-(1-1/p)^p) \leq Y + C - Y = C,$$

and from the last inequality, we obtain

$$g \leq \frac{1}{(1-(1-1/p)^p)} \ C,$$

which proves Claim 2 and establishes Lemma 5.2. ∎

Now we give a proof of Lemma 3.5. This version of the proof is due to Sgall (see the acknowledgments).

LEMMA 3.5. *Suppose we are given numbers* $1 > \Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_m \geq 0$ *and* $\Delta_{m+1} = 0$. *Further, given are positive numbers* $a_1, a_2, \ldots, a_m$ and Y. *Then we have*

$$\min_{j=0,\ldots,m} \left(\sum_{r=1}^{j} a_r + \frac{Y}{1-\Delta_{j+1}} \right) \leq \frac{e}{e-1} \left(\sum_{r=1}^{m} a_r \Delta_r + Y \right). \tag{48}$$

Proof. We use mathematical induction on the size of inequality m. For $m = 0$, the statement trivially holds. Suppose that the lemma was proved for any inequality of size smaller than m. First, consider the case $\Delta_1 \geq \frac{e-1}{e}$. We can write

$$\begin{aligned} \min_{j=0,\ldots,m} \left(\sum_{r=1}^{j} a_r + \frac{Y}{1-\Delta_{j+1}} \right) &\leq \min_{j=1,\ldots,m} \left(\sum_{r=1}^{j} a_r + \frac{Y}{1-\Delta_{j+1}} \right) \\ &= a_1 + \min_{j=1,\ldots,m} \left(\sum_{r=2}^{j} a_r + \frac{Y}{1-\Delta_{j+1}} \right). \end{aligned}$$

The latter minimum is the left-hand side of (48) for a smaller sequence: $\Delta_2, \ldots, \Delta_m$ and $a_2, \ldots, a_m$. Applying the induction hypothesis, we can bound the last expression from above as follows (we also use our bound on Δ_1):

$$\begin{aligned} a_1 + \frac{e}{e-1} \left(\sum_{r=2}^{m} a_r \Delta_r + Y \right) &\leq a_1 \cdot \Delta_1 \frac{e}{e-1} + \frac{e}{e-1} \left(\sum_{r=2}^{m} a_r \Delta_r + Y \right) \\ &= \frac{e}{e-1} \left(\sum_{r=1}^{m} a_r \Delta_r + Y \right). \end{aligned}$$

Thus, we have shown an induction step for the case $\Delta_1 \geq \frac{e-1}{e}$. For the remaining case, $\Delta_1 < \frac{e-1}{e}$, we give a direct proof below.

Suppose $\Delta_1 < \frac{e-1}{e}$. Denote the left-hand side of (48) by X, and notice that

$$\sum_{r=1}^{j} a_r \geq X - \frac{1}{1-\Delta_{j+1}} Y \quad \text{for } 0 \leq j \leq m. \tag{49}$$

The following steps are justified below:

$$\begin{aligned} \sum_{r=1}^{m} a_r \Delta_r + Y &= \sum_{j=1}^{m} \left((\Delta_j - \Delta_{j+1}) \sum_{r=1}^{j} a_r \right) + Y \\ &\geq^{(1)} \sum_{j=1}^{m} (\Delta_j - \Delta_{j+1}) X - \left(\sum_{j=1}^{m} \frac{\Delta_j - \Delta_{j+1}}{1 - \Delta_{j+1}} \right) Y + Y \\ &= \Delta_1 X - \left(\sum_{j=1}^{m} \left(\frac{\Delta_j - 1}{1 - \Delta_{j+1}} + 1 \right) \right) Y + Y \end{aligned}$$

$$= \Delta_1 X + \left(1 - m + \sum_{j=1}^{m} \frac{1-\Delta_j}{1-\Delta_{j+1}}\right) Y$$

$$\geq^{(2)} \Delta_1 X + \left(1 - m + m(1-\Delta_1)^{\frac{1}{m}}\right) Y$$

$$\geq^{(3)} \Delta_1 X + \left(1 - m + m(1-\Delta_1)^{\frac{1}{m}}\right)(1-\Delta_1) X$$

$$= \left(1 + m(-1 + (1-\Delta_1)^{\frac{1}{m}})(1-\Delta_1)\right) X \geq^{(4)} \left(1 - \frac{1}{e}\right) X.$$

(1) Here we use the ordering of the deltas and inequality (49).

(2) This inequality follows from the arithmetic-geometric mean inequality $\sum_{j=1}^{m} x_j \geq m(\prod_{j=1}^{m} x_j)^{1/m}$ used for positive numbers $x_j = \frac{1-\Delta_j}{1-\Delta_{j+1}}$.

(3) Here we use inequality $Y \geq (1-\Delta_1)X$, which is implied by (49) for $j = 0$ and the fact that the coefficient of Y is nonnegative, which follows from $1 - \Delta_1 \geq \frac{1}{e} \geq (1 - \frac{1}{m})^m$.

(4) This inequality is elementary calculus: The minimum of the left-hand side over all Δ_1 is achieved for $1 - \Delta_1 = (\frac{m}{m+1})^m$, and, after substituting this value, it reduces to $1 - (\frac{m}{m+1})^{m+1} \geq 1 - \frac{1}{e}$. ∎

Lemma 5.3. *Let $P(m) \in \mathbb{N}$ be defined as follows:*

$$\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1} \leq 1 \quad \textit{and} \tag{50}$$

$$\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1} + \frac{1}{P(m)} \geq 1. \tag{51}$$

Then,

$$\lim_{m\to\infty} \frac{m}{m - P(m)(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1})} = \frac{e}{e-1}.$$

Proof. Let us first find $\lim_{m\to\infty} P(m)/m$. Observe that the following inequalities hold:

$$\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1} \geq \int_{P(m)+1}^{m+1} \frac{1}{x}\, dx = \ln \frac{m+1}{P(m)+1},$$

$$\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)} \leq \int_{P(m)-1}^{m} \frac{1}{x}\, dx = \ln \frac{m}{P(m)-1}$$

(the equalities follow from $\int_a^b 1/x\, dx = \ln b/a$). Then (50) and (51) imply

$$1 \geq \ln \frac{m+1}{P(m)+1}, \qquad 1 \leq \ln \frac{m}{P(m)-1}.$$

From this we have

$$\frac{m+1}{e} - 1 \leq P(m) \leq \frac{m}{e} + 1.$$

Dividing by m,

$$\frac{1+1/m}{e} - 1/m \leq \frac{P(m)}{m} \leq \frac{1}{e} + 1/m.$$

Now we see that $\lim_{m\to\infty} P(m)/m = 1/e$.

Let us now find $\lim_{m\to\infty}\left(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1}\right)$. From (50) and (51) we have

$$1 - \frac{1}{P(m)} \leq \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1} \leq 1.$$

Since we already know that $\lim_{m\to\infty} P(m) = \infty$, we have

$$\lim_{m\to\infty}\left(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1}\right) = 1.$$

Now consider

$$\frac{m}{m - P(m)(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1})} = \frac{1}{1 - \frac{P(m)}{m}(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1})}.$$

Using $\lim_{m\to\infty}\frac{P(m)}{m} = 1/e$ and $\lim_{m\to\infty}\left(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1}\right) = 1$ we have

$$\lim_{m\to\infty}\frac{1}{1 - \frac{P(m)}{m}(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{P(m)+1})} = \frac{1}{1-1/e} = \frac{e}{e-1},$$

which establishes the lemma. ∎

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