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Lifting theorems and facet characterization for a class of clique partitioning inequalities

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Abstract

In this paper we prove two lifting theorems for the clique partitioning polytope, which provide sufficient conditions for a valid inequality to be facet-defining. In particular, if a valid inequality defines a facet of the polytope corresponding to the complete graph K_m on m vertices, it defines a facet for the polytope corresponding to K_n for all $n > m$. This answers a question raised by Grötschel and Wakabayashi. Further, for the case of arbitrary graphs, we characterize when the so-called 2-partition inequalities define facets. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In order to solve an integer linear program by a cutting plane algorithm, it is desirable to know the convex hull of the feasible solutions to the problem in question. When a valid inequality defines a facet then at least for some objective function this inequality is necessary in obtaining an integral solution (see [7] for a general introduction to polyhedral theory).

In this paper we prove two theorems of the following kind: if a valid inequality for the clique partitioning polytope (see below) satisfies a certain set of conditions, then this inequality defines a facet of this polytope.

A consequence of one of these theorems is that if a valid inequality defines a facet of the polytope corresponding to the graph K_m, i.e. the complete graph on m nodes, it also defines a facet for the polytope corresponding to K_n for all $n > m$. We further show that if one is able to ‘cover’ in a certain way the support of a given valid inequality with facets, then the inequality is facet-defining. This result allows us, in some cases, to simplify proofs that inequalities define facets.

This section proceeds with describing the clique partitioning problem, related literature, and some notation. In Section 2 we state our main results, and in Section 3 we apply these results by characterizing

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when (generalized) 2-partition inequalities define facets in the case of arbitrary graphs.

1.1. The problem

The clique partitioning problem can be described as follows. Let $G=(V,E)$ be a (not necessarily complete) graph with edge weights $w_{ij}\in\mathbb{R}$ for all $\{i,j\}\in E$. A subset $D\subseteq E$ of the edges is called a *clique partition* if the components of the graph $H=(V,D)$ are cliques, i.e. D induces a partition of the vertex set V. The problem is to find a clique partition of maximal weight.

Mathematically, the clique partitioning problem can be formulated as follows:

$$\begin{aligned}
&\text{Maximize} && \sum_{\{i,j\}\in E} w_{ij}x_{ij} \\
&\text{subject to} && x_{ij}+x_{ik}-x_{jk}\leqslant 1 \quad \forall i,j,k\in V, \\
& && \quad \{i,j\},\{i,k\},\{j,k\}\in E \qquad (1)\\
& && x_{ij}+x_{ik}\leqslant 1 \quad \forall i,j,k\in V, \\
& && \quad \{i,j\},\{i,k\}\in E,\{j,k\}\notin E \\
& && x_{ij}\in\{0,1\} \quad \forall\{i,j\}\in E
\end{aligned}$$

where $x_{ij}=1$ if $\{i,j\}$ belongs to the clique partition, and $x_{ij}=0$ otherwise. The clique partitioning polytope $P(G)$ is defined as the convex hull of all feasible solutions to (1). The inequalities $x_{ij}+x_{ik}-x_{jk}\leqslant 1$ are called *triangle inequalities*. The inequalities $x_{ij}\geqslant 0$ and $x_{ij}\leqslant 1$ in the LP relaxation of (1) are called *trivial inequalities*. Whenever any of these inequalities define facets, those are called *trivial facets*. The nonnegativity constraints always define facets, the upper bound constraint $x_{ij}\leqslant 1$ defines a facet if and only if $\{i,j\}$ is an isolated edge (a component of G). We will henceforth assume that $P(G)$ has nontrivial facets, that is, some component of G has at least three vertices. Every nontrivial facet-defining inequality of $P(G)$ written in the form $d^{\mathrm{T}}x\leqslant d_0$ necessarily satisfies $d_0>0$.

1.2. Related literature

For complete graphs G, the polytope $P(G)$ is studied in [4,5,8]. Closely related to $P(G)$ are polytopes investigated in [1–3].

1.3. Notation

Let $S,T\subseteq V$, $D\subseteq E$, and let $a^{\mathrm{T}}x\leqslant a_0$ be a valid inequality for $P(G)$. The set of edges in G with one node in S and the other in T is denoted by $E(S,T)$, i.e.

$$E(S,T):=\{\{i,j\}\in E\mid i\in S,\ j\in T\}.$$

We write $E(S)$ for $E(S,S)$. Furthermore, $\chi^D\in\{0,1\}^{|E|}$ denotes the incidence vector (also called characteristic vector) of D, i.e. $\chi^D_{ij}=1$ if $\{i,j\}\in D$, and $\chi^D_{ij}=0$ otherwise. $V(D)$ denotes the set of vertices incident to an edge in D. The vector $a_D\in\mathbb{R}^{|E|}$ is defined by

$$(a_D)_{ij}:=\begin{cases} a_{ij} & \text{if } \{i,j\}\in D,\\ 0 & \text{otherwise}\end{cases}$$

and the truncated vector $\bar{a}_D\in\mathbb{R}^{|D|}$ is defined by $\bar{a}_{ij}:=a_{ij}$ for all $\{i,j\}\in D$.

The *support* of the inequality $a^{\mathrm{T}}x\leqslant a_0$ is the subset $E_a:=\{\{i,j\}\in E\mid a_{ij}\neq 0\}$ of E.

Finally, we use the following shorthand: given a graph $G=(V,E)$, let

$$x(F):=\sum_{\{i,j\}\in F} x_{ij} \quad \text{for } x\in P(G) \text{ and } F\subseteq E.$$

2. Two lifting theorems

Given a polytope P and an inequality valid for P, a natural question to consider is whether this inequality defines a facet of P. For the clique partitioning polytope Grötschel and Wakabayashi [4] formulated some necessary conditions for a valid inequality to be facet-defining. For instance, they show that the support of any nontrivial facet-defining inequality must be 2-connected.

In each of the next two subsections we formulate a theorem stating sufficient conditions for a valid inequality for $P(G)$ to be facet-defining.

2.1. Once a facet, always a facet

One way to find facet-defining inequalities of $P(G)$ is by considering the facial structure of the clique partitioning polytope of a subgraph of G. This idea is used in the next theorem, which comes down to the following: if $G_0=(V_0,E_0)$ is a subgraph of $G=(V,E)$, and if $\{\{u,v\}\mid u\in V_0,\ v\in V\backslash V_0\}\subseteq E$, then any facet-defining inequality of $P(G_0)$ can be extended in a trivial way into a facet-defining inequality of $P(G)$.

Theorem 2.1. *Let $G=(V,E)$ be a graph and $H=(U,D)$ be an induced subgraph of G. If*
(i) *$\bar{a}_D^{\mathrm{T}}\bar{x}_D \leqslant a_0$ defines a nontrivial facet of $P(H)$, and*
(ii) *$E(U,V\backslash U)\subseteq E$,*
then $a^{\mathrm{T}}x\leqslant a_0$, where $a^{\mathrm{T}}=(\bar{a}_D^{\mathrm{T}},0^{\mathrm{T}})$, defines a facet of $P(G)$.

Proof. It is immediate that $a^{\mathrm{T}}x\leqslant a_0$ is valid for $P(G)$. To show that it is facet defining, consider the following. Given the facet $\bar{F}=\{\bar{x}\in P(H)\mid \bar{a}^{\mathrm{T}}\bar{x}=a_0\}$ of $P(H)$, the lifted face $F=\{x\in P(G)\mid a^{\mathrm{T}}x=a_0\}$ is contained in some facet F_d of $P(G)$, which may be defined by a valid inequality of the form $d^{\mathrm{T}}x\leqslant a_0$. The clique partitions of H can be regarded as clique partitions of G not containing any of the edges in $E\backslash D$. The corresponding characteristic vectors $\bar{x}\in P(H)$ are thereby turned into expanded vectors $x^{\mathrm{T}}=(\bar{x}^{\mathrm{T}},0^{\mathrm{T}})$ in $P(G)$ by adding a zero vector of length $|E\backslash D|$. Writing $d^{\mathrm{T}}=(b^{\mathrm{T}},c^{\mathrm{T}})$, where c is a vector again of that length, we then obtain

$$\bar{a}^{\mathrm{T}}\bar{x}=a_0=d^{\mathrm{T}}x=b^{\mathrm{T}}\bar{x}\quad \forall \bar{x}\in\bar{F}$$

and consequently $b=\bar{a}$. Hence it remains to show that $c=0$. It is obvious that $c_{ij}=0$ for all $\{i,j\}\in E(V\backslash U)$. Now, let $\bar{x}\in\bar{F}$ be the characteristic vector of a clique partition, and let $S_1,\ldots,S_r\subseteq U$ be the partition of U induced by $\bar{x}$. For all $k=1,\ldots,r$ let v^k be the characteristic vector of length $|U|$ of S_k, i.e. $v_i^k=1$ if $i\in S_k$ and $v_i^k=0$ otherwise. Let $u\in V\backslash U$. The clique partition $\bar{x}$ can be extended to r distinct clique partitions of G by adjoining vertex u to exactly one of the sets S_k $(k=1,\ldots,r)$. (This is possible due to condition (ii)). Their characteristic vectors $y^1,\ldots,y^r$, as well as x, all belong to the lifted face F (since $V(E_a)\subseteq U$), and thus also to the facet F_d of $P(G)$, whence

$$\sum_{i\in S_k} c_{ui}=d^{\mathrm{T}}(y^k-x)=0\quad \forall k=1,\ldots,r,$$

or equivalently

$$c_u^{\mathrm{T}}v^k=0\quad \forall k=1,\ldots,r,$$

where the vector c_u is indexed by the vertices of U. Let $\bar{x}^k=\chi^{E(S_k)}$ $(k=1,\ldots,r)$, so that $\bar{x}=\bar{x}^1+\cdots+\bar{x}^r$, and let A denote the vertex-edge incidence matrix of the graph H. We obtain

$$A\bar{x}^k=(|S_k|-1)\cdot v^k\quad \forall k=1,\ldots,r$$

and therefore

$$c_u^{\mathrm{T}}A\bar{x}^k=(|S_k|-1)\cdot c_u^{\mathrm{T}}v^k=0\quad \forall k=1,\ldots,r.$$

Since $\bar{F}$ is a nontrivial facet of $P(H)$, there exist $m:=|D|$ linearly independent vectors $w_i\in\bar{F}$ of clique partitions of H $(i=1,\ldots,m)$. Each w_i induces a partition of U into $S_i^1,\ldots,S_i^{r_i}$. Let $w_i^j:=\chi^{E(S_i^j)}$ $(i=1,\ldots,m,$ $j=1,\ldots,r_m)$, and define the matrix W as

$$W=[w_1^1\ldots w_1^{r_1}|\ldots\ldots|w_m^1\ldots w_m^{r_m}].$$

Then W satisfies $c_u^{\mathrm{T}}AW=0$. The matrix W has rank m since $w_1,\ldots,w_m$ belong to the linear space generated by the columns of W. As for the rank of A, we use the following lemma.

Lemma 2.2. *Let $H=(U,E)$ be a connected graph, and let A be the vertex-edge incidence matrix of H. If H is bipartite, then A has rank $|U|-1$, otherwise A has rank $|U|$.*

The proof of this fact is as follows. For all $e\in E$ denote by a_e the column of A indexed by e. The vertex-edge incidence matrix of any spanning tree of H has rank $|U|-1$, so $\operatorname{rank}(A)\geqslant|U|-1$. Two cases can occur.

Case 1: H is bipartite. Let $T=(U,D)$ be a spanning tree of H, and consider any edge $e_1\in E\backslash D$. The graph $(H,D\cup\{e_1\})$ contains an even cycle $\{e_1,\ldots,e_k\}$, and

$$\sum_{i=1}^{k}(-1)^k\cdot a_{e_i}=0.$$

Hence a_{e_1} is a linear combination of $a_{e_2},\ldots,a_{e_k}$.

Case 2: H is not bipartite. Then H contains an odd cycle $\{e_1,\ldots,e_k\}$. Choose a spanning tree $T=(U,D)$ of H containing the edges $e_2,\ldots,e_k$, but not containing edge e_1. The vertex-edge incidence matrix of T has rank $|U|-1$, and a_{e_1} is not a linear combination of $a_{e_2},\ldots,a_{e_k}$ (compare [6]). Hence a_{e_1} is not a linear combination of the columns associated to the edge set D of T (as is seen by a straightforward inductive argument). Therefore A has rank $|U|$, which proves the lemma.

We now proceed with the proof of Theorem 2.1 by distinguishing two cases.

Case 1: The graph H contains an odd cycle. From Lemma 2.2 it follows that A has rank $|U|$, so the product matrix AW has full row rank, because both A and W have full row rank. We therefore conclude that $c_u = 0$, and hence $c = 0$.

Case 2: The graph H contains no odd cycle. Then H is bipartite, and hence any clique in a clique partition contains at most two vertices. It follows that a clique partition of H is nothing but a matching, or equivalently, $P(H)$ is exactly the matching polytope corresponding to H. Therefore the inequality $\bar{a}^{\mathrm{T}}\bar{x} \leqslant a_0$ is a so-called *degree constraint*

$$\sum_{j:\{j,s\}\in D} x_{sj} \leqslant 1 \tag{2}$$

for some $s \in U$ with degree at least 2 (since these degree constraints are the only nontrivial facets of the matching polytope of a bipartite graph). To proceed, we exhibit $|E|$ affinely independent solutions that satisfy (2) at equality. Since (2) defines a facet of $P(H)$ there exist m affinely independent solutions containing only edges in D. It is easy to verify that this set of solutions together with the characteristic vectors of the following clique partitions is still affinely independent: selecting distinct vertices t and t' adjacent to s in H, consider

$\{\{s,t\},\{v,w\}\}$ for all $\{v,w\} \in E(V\setminus U)$,

$\{\{s,t\},\{u,v\}\}$ for all $u \in U\setminus\{s,t\}$ and $v \in V\setminus U$,

$\{\{s,t'\},\{t,v\}\}$ and $E(\{s,t,v\})$ for all $v \in V\setminus U$.

Since all these solutions lie in the face F, and thus also in F_d, it follows that $d^{\mathrm{T}} = (b^{\mathrm{T}}, c^{\mathrm{T}}) = \beta \cdot (\bar{a}^{\mathrm{T}}, 0^{\mathrm{T}})$ for some $\beta > 0$, and hence $c = 0$. □

The following example serves as an illustration of Theorem 2.1.

Example 2.3. Let G be the graph shown in Fig. 1, and let $x_{12} + x_{13} - x_{23} \leqslant 1$ be the given valid inequality. Since this inequality defines a facet for K_3 (so (i) is fulfilled), and each edge between the node sets $\{1,2,3\}$ and $\{4,5,6\}$ is in E (so (ii) is fulfilled), the inequality $x_{12}+x_{13}-x_{23} \leqslant 1$ is facet-defining for $P(G)$.

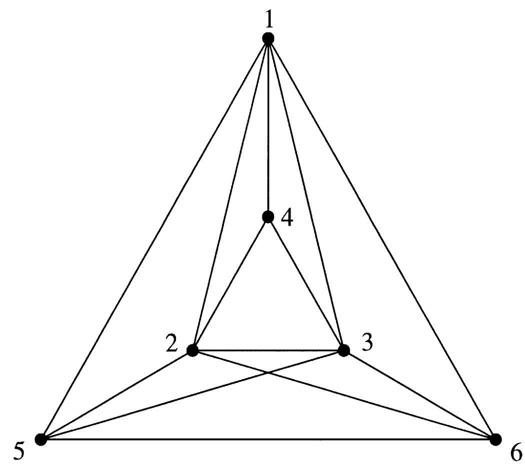

Fig. 1. The graph of Example 2.3.

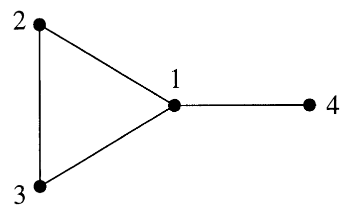

Fig. 2. The graph of Example 2.4.

The relevance of condition (ii) in Theorem 2.1 follows from the following example.

Example 2.4. Let G be the graph shown in Fig. 2.

(a) Let $x_{12} + x_{13} - x_{23} \leqslant 1$ be the given valid inequality. Condition (ii) is not fulfilled, because the edges $\{2,4\}$ and $\{3,4\}$ are missing, and indeed the inequality does not define a facet of $P(G)$ because it is implied by $x_{12} + x_{13} + x_{14} - x_{23} \leqslant 1$.

(b) Let $x_{12} + x_{23} - x_{13} \leqslant 1$ be the given valid inequality. Although condition (ii) is not fulfilled, this inequality defines a facet of $P(G)$.

Example 2.4(b) implies that condition (ii) is not a necessary condition. However, it is always fulfilled in the case of complete graphs. In that case, Theorem 2.1 is equivalent to the following corollary.

Corollary 2.5. *Let $b^{\mathrm{T}}y \leqslant b_0$ be a facet-defining inequality for the polytope $P(K_m)$ $(m \geqslant 3)$. Then for* *every $n>m$, the inequality $a^{\mathrm{T}}x\leqslant a_0=b_0$, where $a^{\mathrm{T}}=(b^{\mathrm{T}},0^{\mathrm{T}})$ and $x^{\mathrm{T}}=(y^{\mathrm{T}},0^{\mathrm{T}})$, defines a facet of $P(K_n)$.*

This corollary confirms that the condition employed by Grötschel and Wakabayashi [4] to ensure that lifted facets remain facets is superfluous.

2.2. Covering a given valid inequality with facets

In this subsection we present our second lifting theorem. Intuitively, the theorem says that if one is able to cover in a certain way the support of a given valid inequality with known facets, then the inequality is facet-defining. The proof is based on the following idea: given a valid inequality $a^{\mathrm{T}}x\leqslant a_0$ and its support E_a, it is possible to extend the set of affinely independent solutions corresponding to some facet-defining inequality "contained" in the support E_a; even more, it is possible to combine the extended sets of affinely independent solutions of different facets contained in the support E_a to show that the given inequality defines a facet.

In order to give a precise formulation of the lifting theorem, we need some additional notation. Given the graph $G=(V,E)$, we define, for $D\subseteq E$, $X_G(D)$ by

$$X_G(D) := \{x\in P(G)\cap\{0,1\}^{|E|}\mid x_{ij}=0 \quad \forall\{i,j\}\in E\backslash D \text{ with } \{i,j\}\cap V(D)\neq\emptyset\}.$$

In words, $X_G(D)$ is the set of clique partitions such that each clique has either all its edges in D or all its edges in the graph induced by the vertex set $V\backslash V(D)$.

The following example illustrates this concept.

Example 2.6. Let K_5 be the complete graph on 5 vertices, and let $D=\{\{1,2\},\{1,5\}\}$ (see Fig. 3). Then the only clique partitions whose characteristic vectors belong to $X_{K_5}(D)$ are $\emptyset$, $\{\{1,2\}\}$, $\{\{1,5\}\}$, $\{\{3,4\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{1,5\},\{3,4\}\}$.

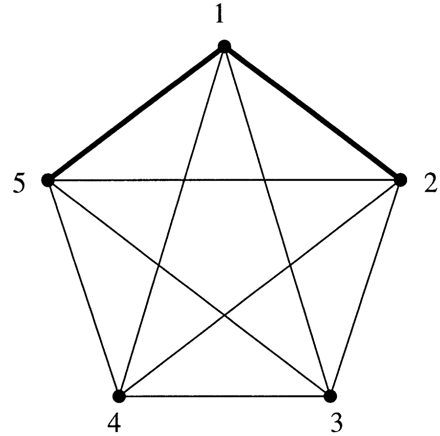

Fig. 3. The graph K_5 and the edge set $D=\{\{1,2\},\{1,5\}\}$.

Next, we define the *load* and the *residual* of D with respect to $a^{\mathrm{T}}x\leqslant a_0$ by

$$\mathrm{load}_G(a,D):=\max\{a_D^{\mathrm{T}}x\mid x\in X_G(D)\},$$

$$\mathrm{res}_G(a,D):=\max\{(a-a_D)^{\mathrm{T}}x\mid x\in X_G(D)\}.$$

Notice that both the load and the residual are always nonnegative since $\chi^{\emptyset}\in X_G(D)$. Let us again illustrate the idea by means of an example.

Example 2.7. Let K_3 be the complete graph with vertex set $\{1,2,3\}$. The load with respect to $a^{\mathrm{T}}x=x_{12}+x_{13}-x_{23}\leqslant 1=a_0$ equals 0 for the clique partitions $\emptyset$ and $\{\{2,3\}\}$, and 1 otherwise; the residual is always 0 except for the empty clique partition.

Notice that the following inequality is valid by definition.

$$\max_{x\in X_G(D)} a_D^{\mathrm{T}}x+\max_{x\in X_G(D)}(a-a_D)^{\mathrm{T}}x = \max_{x\in X_G(D)} a^{\mathrm{T}}x\leqslant \max_{x\in P(G)} a^{\mathrm{T}}x\leqslant a_0.$$

So, $\mathrm{load}_G(a,D)+\mathrm{res}_G(a,D)\leqslant a_0$ for all $D\subseteq E$, and equality holds if and only if there exists a clique partition $x\in X_G(D)$ such that $a^{\mathrm{T}}x=a_0$. In general, establishing the load and the residual of an edge set $D\subseteq E$ with respect to a valid inequality $a^{\mathrm{T}}x\leqslant a_0$ can be quite hard. However, for a specific D it can be relatively easy to find $x,y\in X_G(D)$ with $a_D^{\mathrm{T}}x+(a-a_D)^{\mathrm{T}}y=a_0$, and hence $\mathrm{load}_G(a,D)=a_D^{\mathrm{T}}x$ and $\mathrm{res}_G(a,D)=(a-a_D)^{\mathrm{T}}y$.

We need one more definition to formulate our second result.

Definition 2.8. Let $a^{\mathrm{T}}x\leqslant a_0$ be a valid inequality for $P(G)$, and let $E_1,\ldots,E_k\subseteq E$. Then the *intersection graph* $G(E_1,\ldots,E_k;a)=(U,F)$ is defined by $U=\{1,\ldots,k\}$, and $\{i,j\}\in F$ if and only if there is an edge $e\in E_i\cap E_j$ with $a_e\neq 0$.

Theorem 2.9. *Let $G=(V,E)$ be a graph and $a^{\mathrm{T}}x\leqslant a_0$ a valid inequality for $P(G)$. Further, let $E_1,\dots,E_k\subseteq E(k\geqslant 2)$ be proper subsets of E such that*

(i) $E=\bigcup_{l=1}^{k}E_l$,

(ii) $\mathrm{load}_G(a,E_l)$ *is positive, and* $\bar{a}_{E_l}^{\mathrm{T}}\bar{x}_{E_l}\leqslant \mathrm{load}_G(a,E_l)$ *defines a facet of* $P(V,E_l)$ *for each* $l=1,\dots,k$,

(iii) $\mathrm{load}_G(a,E_l)+\mathrm{res}_G(a,E_l)=a_0$, *for all* $l=1,\dots,k$,

(iv) *the intersection graph* $G(E_1,\dots,E_k;\ a)$ *is connected.*

Then the inequality $a^{\mathrm{T}}x\leqslant a_0$ *defines a nontrivial facet of* $P(G)$.

Proof. The face $F_a:=\{x\in P(G)\,|\,a^{\mathrm{T}}x=a_0\}$ is contained in the facet F_d defined by some valid inequality $d^{\mathrm{T}}x\leqslant d_0$. We wish to show that $d=\beta\cdot a$ for some $\beta>0$. For each $l=1,\dots,k$ choose a maximal set M_l of affinely independent integral points $x=x_{E_l}$ satisfying $\bar{x}_{E_l}\in P(V,E_l)$ and $a_{E_l}^{\mathrm{T}}x=\mathrm{load}_G(a,E_l)$. Since none of the facets described in condition (ii) contains the zero vector, each set M_l is linear independent. According to condition (iii), we can find a vector $y^l\in X_G(E_l)$ with $(a-a_{E_l})^{\mathrm{T}}y^l=\mathrm{res}_G(a,E_l)$ for each $l=1,\dots,k$, thus yielding

$$
\begin{aligned}
a^{\mathrm{T}}(x+y^l_{E\setminus E_l}) &= a_{E_l}^{\mathrm{T}}x+(a-a_{E_l})^{\mathrm{T}}y^l\\
&= \mathrm{load}_G(a,E_l)+\mathrm{res}_G(a,E_l)\\
&= a_0 \quad \forall x\in M_l \quad (l=1,\dots,k).
\end{aligned}
$$

It follows that

$$\{x+y^l_{E\setminus E_l}\,|\,x\in M_l\}\subseteq X_G(E_l)\cap F_a \quad \text{for}\ \ l=1,\dots,k.$$

Put

$$\beta_l:=\frac{d_0-d^{\mathrm{T}}_{E\setminus E_l}y^l}{\mathrm{load}_G(a,E_l)}>0,$$

then

$$(d-\beta_l a)^{\mathrm{T}}_{E_l}x=0 \quad \forall x\in M_l \quad (l=1,\dots,k).$$

Since M_l is a linearly independent set of size $|E_l|$, we conclude that $d_{E_l}=\beta_l a_{E_l}$ $(l=1,\dots,k)$.

If l and m are adjacent vertices of the intersection graph $G(E_1,\dots,E_k;a)$, then $a_e\neq 0$ for some $e\in E_l\cap E_m$ whence $\beta_l a_e=d_e=\beta_m a_e$, that is, $\beta_l=\beta_m$. Since the intersection graph is connected by condition (iv), we obtain

$$\beta:=\beta_1=\beta_2=\cdots=\beta_k.$$

As $E_1,\dots,E_k$ cover E by condition (i), we infer that $d=\beta\cdot a$ and $d_0=\beta\cdot a_0$. Therefore $a^{\mathrm{T}}x\leqslant a_0$ defines the facet F_d of $P(G)$. □

Notice that, given the edge sets $E_1,\dots,E_k$, and provided that the load and residual of each edge set is known, it is easy to check whether conditions (i)–(iv) of Theorem 2.9 are fulfilled.

Before deriving a corollary from Theorem 2.9, let us first illustrate its use.

Example 2.10. Let K_5 be the complete graph with vertex set $\{1,2,3,4,5\}$. The 2-chorded-cycle inequality (see [4])

$$
\begin{aligned}
&x_{12}+x_{23}+x_{34}+x_{45}+x_{51}-x_{13}-x_{24}\\
&\quad -x_{35}-x_{41}-x_{52}\leqslant 2 \qquad (3)
\end{aligned}
$$

can be covered by triangle inequalities in the sense of Theorem 2.9: the five sets

$$E_l=E(\{l,l+1,l+2\})$$

$$\text{(numbers modulo } 5, 1\leqslant l\leqslant 5)$$

meet all requirements.

In fact, inequality (3) defines a facet for any polytope corresponding to a graph containing K_5 as a subgraph, which is an immediate consequence of the following corollary:

Corollary 2.11. *Let $G=(V,E)$ be a graph, $H=(U,D)$ an induced subgraph of G, and $a^{\mathrm{T}}x\leqslant a_0$ a valid inequality for $P(G)$. Further, let D include the support of $a^{\mathrm{T}}x\leqslant a_0$ and the following two conditions hold*:

(i) $\bar{a}_D^{\mathrm{T}}\bar{x}_D\leqslant a_0$ *defines a facet of* $P(H)$,

(ii) *for each $u\in U$ incident to an edge between U and $V\setminus U$ there exists some edge $\{i,j\}\in E(U\setminus\{u\})$ with*

$$\mathrm{load}_G(a,E(U\setminus\{u,i,j\}))=a_0-a_{ij}<a_0.$$

Then $a^{\mathrm{T}}x\leqslant a_0$ defines a facet of $P(G)$.

Proof. For each vertex u of U choose an edge $e(u)$ as described in (ii). Put

$$E_u:=E(\{u\},V\setminus U)\cup E(V\setminus U)\cup\{e(u)\}.$$

Then the sets E_u ($u \in U$) together with D meet the requirements of Theorem 2.9. □

Notice that given a complete graph on $2k+1$ vertices ($k \geqslant 3$) and a 2-chorded-cycle inequality with right-hand side k, the construction shown in Example 2.10 fails to cover all edges of G and hence Theorem 2.9 (or Corollary 2.11) alone does not imply facetness of these inequalities for polytopes corresponding to complete graphs (or graphs containing complete subgraphs). More generally, invoking Theorem 2.9 for proving that a given inequality is facet-defining for the polytope $P(V,E)$ would sometimes only confirm that the "truncated" inequality is facet-defining for $P(V,D)$ where D includes the support but is a proper subset of E. In such circumstances a straightforward construction may turn the base of $\mathbb{R}^{|D|}$ provided by Theorem 2.9 into an expanded base of $\mathbb{R}^{|E|}$ that resides within the face in question, thus proving facetness:

Lemma 2.12. *Let $G=(V,E)$ be a graph and $a^{\mathrm{T}}x \leqslant a_0$ a valid inequality for $P(G)$. Let D be a proper subset of E such that $\bar{a}_D^{\mathrm{T}}\bar{x}_D \leqslant a_0$ defines a nontrivial facet of $P(V,D)$.* Let *$F_1,\dots,F_k$ be a covering of $E \backslash D$ such that for each $i=1,\dots,k$ there exist $|F_i|$ solutions x of $a^{\mathrm{T}}x=a_0$ on $D \cup (\bigcup_{j=1}^{i} F_i)$ for which the corresponding truncated vectors $\bar{x}_{F_i}$ are linearly independent. Then $a^{\mathrm{T}}x \leqslant a_0$ defines a facet of $P(G)$.*

3. General 2-partition inequalities

For disjoint sets S,T of V with $1 \leqslant |S| \leqslant |T|$

$$x(E(S,T)) - x(E(S)) - x(E(T)) \leqslant |S| \qquad (4)$$

is called a *general* 2-*partition inequality*, or an (S,T)-*inequality* for short. For complete graphs G, this inequality is valid for $P(G)$ as was shown in [4]; here is an alternative (shorter) argument showing validness for arbitrary graphs: let x be the characteristic vector of a clique partition, and let $U_1,\dots,U_r$ be the sets of vertices incident to its nontrivial cliques. Intersected with S and T, respectively, each set U_i splits into two subsets, one with α_i elements and another with β_i elements, where $\alpha_i + \beta_i = |U_i|$ and $0 \leqslant \alpha_i \leqslant \beta_i$ ($i=1,\dots,r$). Since the inequalities

$$(\alpha_i - \beta_i)(\alpha_i - \beta_i + 1) \geqslant 0 \quad (i=1,\dots,r)$$

trivially hold (because only integers are involved), so do the inequalities

$$\alpha_i\beta_i - \binom{\alpha_i}{2} - \binom{\beta_i}{2} \leqslant \alpha_i \quad (i=1,\dots,r).$$

Summing up these inequalities we obtain that the left-hand side of (4) for the particular choice of x is no larger than $\sum_i \alpha_i \leqslant |S|$, as required.

Not every (S,T)-inequality is actually facet-defining: consider for instance $S=\{1\}$ and $T=\{2,3\}$ in Example 2.4a. We will now characterize when such an inequality defines a facet under the additional hypothesis that $E(S,T)$ has full size $|S| \cdot |T|$.

Theorem 3.1. *Given a graph $G=(V,E)$, let S and T be disjoint subsets of V such that $1 \leqslant |S| \leqslant |T|$ and $\{\{s,t\} \mid s \in S, t \in T\} \subseteq E$.*

Then the (S,T)-inequality defines a facet of $P(G)$ if and only if $|S| < |T|$ and the following two conditions hold:

(i) *if $|S| > 1$, then $E(T) \neq \emptyset$,*

(ii) *every vertex of G outside $S \cup T$ which is adjacent to some vertex in S is also adjacent to some vertex in T.*

Proof. First we show that the conditions are necessary. If $|S|=|T|$, then every clique partition for which the characteristic vector belongs to the face F supported by (4) consists of cliques that intersect S and T in equally sized subsets. Therefore all members of F satisfy the additional equality $x(S)-x(T)=0$, whence F cannot be a facet.

If $2 \leqslant |S| < |T|$ but $E(T)=\emptyset$, then every clique partition with characteristic vector $x \in F$ must be a matching, so that again $x(S)-x(T)=0$ holds.

If there is some edge $e=\{s,v\} \in E$ with $s \in S$, $v \notin S \cup T$, and $\{v,t\} \notin E$ for all $t \in T$, then we may add x_e to the left-hand side of (4) and still obtain an inequality valid for $P(V,E)$. Hence F is not a facet.

Conversely, assume that all requirements of the theorem are met. For each triplet s,t,u with $s \in S$ and $t,u \in T$ we take the edge set $E(\{s,t,u\})$. These edge sets form a connected intersection graph with respect to (4) since S is a singleton or $E(T)$ is nonempty (or both). From Theorem 2.9 we infer that the truncated inequality is facet-defining for $P(V,E(S,T) \cup E(T))$. An application of Corollary 2.11 then shows that this is also true for the higher dimensional polytope $P(V,D)$

where

$$D = E(V\backslash T, T) \cup E(T) \cup E(V\backslash(S \cup T))$$
$$= E\backslash(E(S, V\backslash T)).$$

Each edge $e \in E\backslash D$ is extended to a clique: if $e \in E(S)$ then the clique $E(e\cup f)$ is constructed where $f \in E(T)$ (which exists by condition (i)); else a triangle $E(e \cup \{t\})$ is constructed where $t \in T$ is chosen appropriately (this is possible by condition (ii)). Each of these cliques can be extended to a clique partitioning of (V, E) that otherwise uses only single edges between S and T as cliques. These $|E\backslash D|$ clique partitionings satisfy the conditions of Lemma 2.12. This ensures that the (S, T)-inequality is indeed facet-defining for $P(V, E)$. □

Theorem 2.9 can also be used to construct relatively short proofs that inequalities define facets, as is illustrated in the next example. This entails, as a particular instance, the fact that the general 2-partition inequalities considered by Grötschel and Wakabayashi [5] define facets of $P(G)$.

Example 3.2. Let $G = (V, E)$ be a complete graph, and let $S_1, \dots, S_k$ be pairwise disjoint subsets of V which are disjoint from subsets $T_1, \dots, T_k$ of $V (k \geqslant 2)$, satisfying the following additional conditions. For any pairwise intersecting sets selected from $T_1, \dots, T_k$, all pairwise intersections are equal and contain at least two vertices. The intersection graph of the sets $T_1, \dots, T_k$ is a block graph, that is, a graph in which every maximal 2-connected subgraph is complete. Moreover,

$$1 \leqslant |S_i| \leqslant |T_i| - \max_{\substack{j=1,\dots,k,\\ j\neq i}} |T_i \cap T_j| \quad \text{for all } i = 1, \dots, k.$$

Then the inequality

$$\sum_{i=1}^{k} x(E(S_i, T_i)) - \sum_{i=1}^{k} x(E(S_i)) - x\left(\bigcup_{i=1}^{k} E(T_i)\right) - \sum_{\substack{1\leqslant i<j\leqslant k,\\ T_i\cap T_j\neq\emptyset}} x(E(S_i, S_j)) \leqslant \sum_{i=1}^{k} |S_i| \quad (5)$$

is valid and facet-defining for $P(G)$.

Notice that the general 2-partition inequality (see [5]) simply is inequality (5) under the constraint that the intersection graph of the sets $T_1, \dots, T_k$ is a path.

To show that inequality (5) is valid, proceed by induction. Validity does not require the condition that nonempty intersections of sets from $T_1, \dots, T_k$ have size at least 2. Without loss of generality assume that T_k is a terminal set in the sense that every nonempty intersection with any other of the sets $T_1, \dots, T_{k-1}$ equals some subset U. Let M be any clique partitioning of G. Denote by U' the (possibly empty) subset of U comprising all those $u \in U$ for which M includes an edge $\{s'_u, u\}$ with $s'_u \in S_k$ and an edge $\{s''_u, u\}$ with $s''_u \in \bigcup_{i=1}^{k-1} S_i$. Consider any partition of U' into two sets U_1 and U_2, so $U' = U_1 + U_2$. Put $S' := S_k$, $S'' := \bigcup_{i=1}^{k-1} S_i$, $T' := T_k - U_2$, and $T'' := \bigcup_{i=1}^{k-1} T_i - U_1$. Let M' and M'' be the traces of the clique partitioning M on $S' \cup T'$ and $S'' \cup T''$. Truncating the characteristic vector x of M to $E(S' \cup T')$ and $E(S'' \cup T'')$, respectively, turns (5) into two valid inequalities (5') and (5"), the right hand sides of which add up to $\sum_i |S_i|$. The only edges of M with a positive count in the left hand side of (5) which do not belong to either $E(S' \cup T')$ or $E(S'' \cup T'')$ are of the form $\{t, u\}$ for $t \in S''$, $u \in U_1$ and $\{s, v\}$ for $s \in S'$, $v \in U_2$. The edges of M with a negative count in the left hand side of (5) not belonging to either $E(S' \cup T')$ or $E(S'' \cup T'')$ are of the form $\{t, s\}$ for $t \in S''$, $s \in S'$ and $\{u, v\}$ for $u \in U_1$, $v \in U_2$. We claim that, given sets S', S'' and U', there always exists a partition of U' into U_1 and U_2 such that $|S'||U_2| + |S''||U_1| \leqslant |U_1||U_2| + |S'||S''|$, which implies that the positive count of edges not in $E(S' \cup T') \cup E(S'' \cup T'')$ is compensated by the negative count of edges not in $E(S' \cup T') \cup E(S'' \cup T'')$. Indeed, the latter inequality is equivalent to $(|S'| - |U_1|)(|U_2| - |S''|) \leqslant 0$, which can be achieved when U_1 consists of $\lceil |U'||S'|/(|S'| + |S''|) \rceil$ elements of U' (chosen arbitrarily) and U_2 includes the remaining elements. Therefore the sum of the left-hand sides of (5') and (5") is an upper bound for the left-hand side of (5), thus proving that (5) is a valid inequality for $P(G)$.

Let us now present a proof of the 'facetness' of (5) based on Theorem 2.9 and Lemma 2.12. Put $E_i := E(S_i \cup T_i)$ for $i = 1, \dots, k$. For each index i the load of inequality (5) with respect to E_i equals $|S_i|$

and the corresponding residual equals the sum of all $|S_j|$ with $j \neq i$. Hence all four requirements of Theorem 2.9 are met for the subgraph (V, D) of G where $D := \bigcup_i E_i$. Therefore the truncation of (5) to the edge set D defines a facet of $P(V, D)$.

Notice that the number of vertices in $T := \bigcup_i T_i$ exceeds the size of $S := \bigcup_i S_i$ by at least 2. Furthermore, any matching in the graph (V, D) comprising $|S|$ edges each joining a vertex in S and a vertex in T constitutes a clique partitioning of G satisfying (5) at equality. We will refer to such a matching as an *S-matching*. It is easy to see that for every pair $\{u, v\} \subseteq T$ there is an S-matching not incident to $\{u, v\}$.

In order to fulfill the requirements of Lemma 2.12, we first exhibit for each edge $\{u, v\} \in E \backslash D$ a clique partitioning which includes $\{u, v\}$ as well as an appropriate S-matching M_{uv}, so that (5) is satisfied with equality.

(1) Assume $\{u, v\} \subseteq T$. Then $\{u, v\} \nsubseteq T_i$ for all i. Pick any S-matching M_{uv} not incident to $\{u, v\}$. Then $M_{uv} \cup \{\{u, v\}\}$ is a clique partitioning satisfying (5) at equality.

(2) Assume $u \in S$ and $v \in T$, say $u \in S_i$ and $v \in T_j$ (where $v \notin T_i$). For any chosen $t_u \in T_i$ there exists an S-matching M_{uv} containing $\{u, t_u\}$ but not incident to v. Then $M_{uv} \cup \{\{u, v\}, \{t_u, v\}\}$ is a clique partitioning satisfying (5) at equality.

(3) Assume $\{u, v\} \subseteq S$ say $u \in S_i$ and $v \in S_j$ (for $j \neq i$). If $T_i \cap T_j \neq \emptyset$, select two distinct vertices t_u and t_v in $T_i \cap T_j$; else pick $t_u \in T_i$ and $t_v \in T_j$ arbitrarily. In either case, we can choose an S-matching M_{uv} containing $\{t_u, u\}$ and $\{t_v, v\}$. Then $M_{uv} \cup \{\{u, v\}, \{t_u, t_v\}, \{u, t_v\}, \{v, t_u\}\}$ is a clique partitioning satisfying (5) at equality.

Now let F_1, F_2, F_3 be the sets of clique partitionings constructed in the preceding steps (1), (2), (3). Clearly, $F_1 \cup F_2 \cup F_3$ covers $E \backslash D$. Since every clique partitioning in F_1, F_2, F_3 contains exactly one edge of $E \backslash D$ intersected with $E(T), E(S, T)$, and $E(S)$ respectively, the conditions in Lemma 2.12 are met for the sets F_1, F_2, F_3. We conclude that (5) defines a facet of $P(G)$. $\square$

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