



Operations Research

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To cite this article:

Fabrice Talla Nobibon, Laurens Cherchye, Yves Crama, Thomas Demuyneck, Bram De Rock, Frits C. R. Spieksma (2016)
Revealed Preference Tests of Collectively Rational Consumption Behavior: Formulations and Algorithms. *Operations Research*
64(6):1197-1216. <http://dx.doi.org/10.1287/opre.2016.1527>

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Revealed Preference Tests of Collectively Rational Consumption Behavior: Formulations and Algorithms

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This paper focuses on revealed preference tests of the collective model of household consumption. We start by showing that the decision problems corresponding to testing collective rationality are NP-complete. This makes the application of these tests problematic for (increasingly available) large(r)-scale data sets. We then present two approaches to overcome this negative result. First, we introduce exact algorithms based on mixed-integer programming (MIP) formulations of the collective rationality tests, which can be usefully applied to medium-sized data sets. Next, we propose simulated annealing heuristics, which allow for efficient testing of the collective model in the case of large data sets. We illustrate our methods by a number of computational experiments based on Dutch labor supply data.

Keywords: revealed preference axioms; rationality; mixed-integer programming; global optimization; simulated annealing.

Subject classifications: utility/preference: theory; networks/graphs: applications; economics.

Area of review: Games, Information, and Networks.

History: Received September 2, 2013; revision received March 12, 2015; accepted May 6, 2016. Published online in

Articles in Advance September 12, 2016.

1. Introduction

The word “economics” stems from the Greek “oikos nomein,” which is literally translated as “running the household.” Households form the cornerstone of our society and, as a result, household consumption decisions drive a huge part of our economy. It is by now well established that the unitary model, which assumes that households behave as single decision makers, is not adequate to describe the behavior of households with multiple members. Therefore, as a more useful alternative, Chiappori (1988, 1992) and Apps and Rees (1988) suggested the collective model of household consumption. This model explicitly recognizes that households consist of multiple individuals (or decision makers) with their own (rational) preferences. It assumes that these individuals engage into intrahousehold bargaining processes that obtain Pareto efficient intrahousehold allocations. It is well documented that this collective model provides a better empirical fit of multi-individual household behavior than the unitary model. See, for example,

Fortin and Lacroix (1997), Browning and Chiappori (1998), Cherchye and Vermeulen (2008) and Cherchye et al. (2008 and 2009).

To verify the empirical adequacy of a particular consumption model, it is important to come up with reliable tests that can be applied to household consumption data. These tests check whether the observed household behavior is consistent with the model predictions; such consistent behavior is then commonly referred to as “rational” in terms of the model subject to testing. Preferably, rationality tests avoid all types of ad-hoc choices. From this perspective, the recently developed revealed preference tests of the collective model are particularly appealing (see Cherchye et al. 2007, 2010, 2011b). These revealed preference tests of the collective model build on early work of Samuelson (1938, 1948), Afriat (1967), Diewert (1973), and Varian (1982), who focused on revealed preference tests of the unitary model. Rather than describing a model’s testable restrictions on observed household behavior in terms of

derivatives of certain unobservable functions (e.g., symmetry of the cross-derivatives of the consumer's cost function), the revealed preference approach defines testable implications in terms of finite systems of inequalities that only involve the household consumption choices that are actually observed (and summarized in terms of prices and quantities). As such, the approach effectively minimizes the risk of specification error. It avoids that the collective model is rejected simply because of a wrongly specified functional structure (rather than a bad empirical fit of the model per se). Interestingly, revealed preference tests not only allow for deciding which model is the most appropriate one to describe the observed household decisions, they can also be used to subsequently recover preferences of individual household members, and to predict household decisions in new situations. Given the very nature of revealed preference tests, such recovery and counterfactual analyses will not be contaminated by ad-hoc functional choices.

The starting result of the current paper is that the revealed preference tests of the collective model are NP-complete. Importantly, this contrasts sharply with existing results for the unitary model, for which polynomial time testing algorithms do exist (see, for example, Varian 1982, Teo and Vohra 2003, and Talla Nobibon et al. 2015). This result fits into the emerging literature that utilizes insights from computational complexity theory in the study of economic-theoretical decision problems. We mention here Gilboa and Zemel (1989), Chu and Halpern (2001), Cechlarova and Hajdukova (2002), Fang et al. (2002), Woeginger (2003), Baron et al. (2004, 2008), Brandt and Fisher (2008), Conitzer and Sandholm (2008), Kalyanaraman and Umans (2008), Procaccia and Rosenschein (2008), Galambos (2009), Hudry (2009), Brandt et al. (2010), Deb (2011), Apesteguia and Ballester (2010), Talla Nobibon and Spieksma (2010), Cherchye et al. (2011a), Echenique et al. (2011b), and Demuynek (2014). Essentially, this trend signals that testing the collective model tests can become very difficult in the case of large data sets. Given this first, negative finding, we subsequently propose two practical approaches to apply revealed preference tests of the collective model to large(r) data sets. First, we develop exact algorithms based on a mixed-integer programming (MIP) formulation of the revealed preference tests. Cherchye et al. (2011b) already introduced similar MIP formulations and showed that these formulations are well suited for meaningfully testing the collective model in practice. As we will discuss below, our new MIP algorithms are—from a computational viewpoint—substantially more attractive than the algorithms originally proposed in Cherchye et al. (2011b). Specifically, we show that these exact MIP algorithms can be very useful for medium-sized data sets (i.e., data sets consisting of up to 100–150 observations). However, given their mixed-integer programming nature, they become less useful for analyzing large-scale data sets. Therefore, as a second approach, we propose simulated annealing heuristics that solve a global optimization formulation of the collective rationality tests.

Our main results are particularly relevant when testing the collective consumption model on large(r)-scale data sets. Importantly, such large data sets are increasingly available, and the use of revealed preference tests to analyze these data has recently been advocated in the literature. A first example concerns large-scale scanner data sets, which contain information on (many) household-level purchases collected at checkout scanners in supermarkets. Echenique et al. (2010, 2011a) convincingly argue that scanner data are obvious candidates for using revealed preference insights to better understand household behavior. A second prime example concerns cross-sectional data gathered for a large number of households (e.g., through national budget surveys). Again, the use of revealed preference tests to such data allows for meaningfully assessing the empirical validity of the collective model (see, for example, Cherchye and Vermeulen 2008). In this study, we will illustrate this last application for a large set of Dutch labor supply data. By using computational experiments, we will demonstrate the practical usefulness of both our newly proposed MIP methods and our specially tailored simulated annealing heuristic.

The rest of our paper unfolds as follows. Section 2 sets the stage by presenting the revealed preference tests that we focus on. Section 3 establishes our NP-completeness result. Section 4 introduces our MIP formulation. Section 5 shows the global optimization formulation, which will then form the basis for the simulated annealing heuristics that we present in Section 6. Section 7 contains our computational experiments. Section 8, finally, concludes.

2. Setting the Stage

We consider households with two decision makers acting in an economy with m commodities or goods. We assume that these goods can only be consumed privately without externalities, as opposed to public goods. (However, the algorithms proposed in this paper can be modified to test rationality of collective households consuming public goods.) *Private* consumption of a good means that its consumption by one household member affects the supply available for the other member; as an example, drinking water can only be consumed privately. Consumption *externalities* refer to the fact that one member gets utility from another member's consumption; for an illustration, consider a wife enjoying her husband's nice clothes. *Public* consumption of a good means that consumption of that good by one member does not affect the supply available for another member, and no one can be excluded from consuming it; as an example, a movie watched by both members of the household is a public good.

We suppose that we have observed n household consumption quantity bundles $q_t := (q_{t,1}, \dots, q_{t,m}) \in \mathbb{R}_+^m$ (nonnegative) with corresponding positive prices $p_t := (p_{t,1}, \dots, p_{t,m}) \in \mathbb{R}_{++}^m$ ($t = 1, \dots, n$). The component $q_{t,i}$ (respectively, $p_{t,i}$), for $i = 1, \dots, m$, corresponds to the

quantity of good i bought by the household (respectively, the unit price of good i) at the time t of observation. Note that the scalar product $p'q$ (where p' denotes the transpose of p) represents the total price of bundle $q \in \mathbb{R}_+^m$ at the prices $p \in \mathbb{R}_{++}^m$. We denote the set of observations by $S := \{(p_t, q_t) : t \in N\}$, where $N := \{1, \dots, n\}$ and we refer to S as the *data set*. For ease of exposition, throughout this paper, we use $t \in N$ to refer to the observation (p_t, q_t) .

For a given observation $t \in N$, a *feasible personalized quantity vector* (x_t^1, x_t^2) is one of the infinitely many feasible split ups of the observed quantity vector q_t into a pair of nonnegative vectors $x_t^1, x_t^2 \in \mathbb{R}_+^m$ such that $q_t = x_t^1 + x_t^2$. The true split up of q_t , which represents the quantity of each good effectively consumed by each member in the household, is unobserved. Given the personalized quantity vectors, we define the *personalized consumption data sets* by $S_1 := \{(p_t, x_t^1) : t \in N\}$ for the first member and $S_2 := \{(p_t, x_t^2) : t \in N\}$ for the second member of the household. The following terminology is well established; see Varian (1982, 2006).

DEFINITION 1. Member $l \in \{1, 2\}$ reveals that he or she *directly prefers* bundle x_s^l over bundle x_t^l if and only if $p_s^l x_s^l \geq p_t^l x_t^l$.

In words, member l reveals that he or she directly prefers bundle x_s^l over bundle x_t^l if l has chosen bundle x_s^l while bundle x_t^l was affordable and could have been chosen. For ease of exposition, when this happens, we simply say that member l directly prefers s over t . Considering the transitive closure of the direct preference relation leads to the next definition.

DEFINITION 2. Member $l \in \{1, 2\}$ *prefers* s over t if and only if there exists a sequence $s_1, s_2, \dots, s_k \in N$, with $s = s_1$ and $t = s_k$, such that l directly prefers s_i over s_{i+1} , for $i = 1, 2, \dots, k - 1$.

These notions are used in the definition of the following well-known axioms of revealed preference, for households with a single decision maker (see, e.g., Varian 1982).

DEFINITION 3 (GARP). A personalized consumption data set S_l ($l = 1, 2$) satisfies the *Generalized Axiom of Revealed Preference* (GARP) if and only if, for all observations s and t , when member l prefers s over t , then $p_t^l x_t^l \leq p_t^l x_s^l$.

DEFINITION 4 (SARP). A personalized consumption data set S_l ($l = 1, 2$) satisfies the *Strong Axiom of Revealed Preference* (SARP) if and only if, for all observations s and t , when $x_s^l \neq x_t^l$ and member l prefers s over t , then $p_t^l x_t^l < p_t^l x_s^l$.

Observe that if S_l satisfies SARP then it also satisfies GARP. One may consider that member l has a “rational” consumption behavior if S_l satisfies one of these axioms. Testing whether S_l satisfies GARP or SARP can be done in time $O(n^2)$, using graph theory algorithms (Varian 1982, Talla Nobibon et al. 2015).

The previous axioms can be extended to collective households, as formalized by the following definitions,

where the prefix 2 indicates that we consider households with two decision makers (Cherchye and Vermeulen 2008, Sabbe 2010, Cherchye et al. 2011b).

DEFINITION 5 (2-GARP). A data set S is consistent with the collective consumption model 2-GARP if and only if there exist feasible personalized quantities (x_t^1, x_t^2) , with $x_t^1 + x_t^2 = q_t$, such that, for each member l , the personalized consumption data set S_l satisfies GARP ($l = 1, 2$).

DEFINITION 6 (2-SARP). A data set S is consistent with the collective consumption model 2-SARP if and only if there exist feasible personalized quantities (x_t^1, x_t^2) , with $x_t^1 + x_t^2 = q_t$, such that, for each member l , the personalized consumption data set S_l satisfies SARP ($l = 1, 2$).

For ease of exposition, when S is consistent with the collective consumption model 2-GARP (respectively, 2-SARP), we simply say that S satisfies 2-GARP (respectively, 2-SARP). Notice that if S satisfies 2-SARP then it also satisfies 2-GARP. Cherchye et al. (2011b) establish that 2-GARP provides a necessary and sufficient condition for collective rationality, in the following sense: 2-GARP implies that the consumption decisions (taken jointly by both members) result in Pareto efficient intrahousehold allocations, whereas any consumption data set S that does not satisfy 2-GARP can be directly interpreted as a Pareto inefficiency (implying collectively irrational behavior).

In practice, the collective consumption models described above may lead to weak tests, because they rely on the assumption that we observe only *aggregate* household consumption. As a consequence, most consumption data sets S satisfy 2-SARP, and hence 2-GARP, and these models may fail to recognize irrational collective behavior. To strengthen the models, existing *assignable information* is often taken into account explicitly (Bourguignon et al. 2009, Chiappori 1988, Cherchye and Vermeulen 2008, Sabbe 2010, Cherchye et al. 2011b). We say that a good is *assignable* if, for at least one member in the household, one has observed a positive quantity of that good consumed by that member, for at least one time t (a common example of assignable good is clothing Bourguignon et al. 2009). For member $l \in \{1, 2\}$ and observation $t \in N$, we denote by q_t^l a known vector of assignable quantities, where we assume that $q_t^1 \geq 0$, $q_t^2 \geq 0$, and $q_t^1 + q_t^2 \leq q_t$ for all $t \in N$. In the presence of assignable information, the data set is now represented as $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$. In the rest of this paper, we rely on this representation of S ; when there is no assignable information, we set $q_t^1 = q_t^2 := \mathbf{0}$ for all $t \in N$, where $\mathbf{0}$ is the zero vector.

The axioms for 2-GARP and 2-SARP immediately extend to the setting with assignable information, provided that we adopt the following definition.

DEFINITION 7. The pair $(x_t^1, x_t^2) \in \mathbb{R}_+^{2m}$ is a pair of *feasible personalized quantities* for the data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$ if and only if $q_t^1 \leq x_t^1$, $q_t^2 \leq x_t^2$, and $q_t = x_t^1 + x_t^2$ for all $t \in N$.

In this paper, we investigate the following computational questions: for a given consumption data set S of a two-member household, (i) does S satisfy 2-GARP? (ii) Does S satisfy 2-SARP? If we are given S_1 and S_2 , then the problems reduce to checking whether the personalized quantity vectors are feasible, and whether S_l satisfies GARP, or SARP, for $l = 1, 2$. These two queries can be answered in polynomial time. However, the global problems are considerably harder. The contributions of this paper include the following:

- (1) A proof that testing 2-GARP, as well as 2-SARP is NP-complete, even when there is no assignable information.
- (2) New MIP formulations of the problems.
- (3) Global optimization formulations and simulated annealing heuristics for their solution.
- (4) Extensive computational experiments on real-world data.

3. Problem Statement

We first give a formal definition of the problems in Section 3.1, and we review existing literature on integer programming models for testing rationality in Section 3.2.

3.1. Problem Description

The problems that we study are formally defined as the following *decision problems*.

Problem 2-GARP

Instance: A data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$, where $N := \{1, \dots, n\}$.

Question: Do there exist $x_t^1, x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1, q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the sets $S_l := \{(p_t, x_t^l) : t \in N\}$ satisfy GARP for $l = 1, 2$?

Problem 2-SARP

Instance: A data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$, where $N := \{1, \dots, n\}$.

Question: Do there exist $x_t^1, x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1, q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the sets $S_l := \{(p_t, x_t^l) : t \in N\}$ satisfy SARP for $l = 1, 2$?

Thus, an instance of 2-GARP (respectively, 2-SARP) is entirely defined by the consumption data set S . Finding out whether an instance satisfies 2-GARP, or 2-SARP is difficult:

THEOREM 1. *Problem 2-GARP and problem 2-SARP are NP-complete, even when there is no assignable information.*

PROOF. See the online Appendix A (available as supplemental material at <https://doi.org/10.1287/opre.2016.1527>).

Theorem 1 implies that any method of which the aim is to solve Problem 2-GARP or 2-SARP is likely to experience sharply increasing running times when the number of observations grows.

If there is no assignable information ($q_t^1 = q_t^2 := \mathbf{0}$, for all t) then any data set $S := \{(p_t, q_t) : t \in N\}$ that satisfies GARP (respectively, SARP) also satisfies 2-GARP (respectively, 2-SARP). Furthermore, any instance with at most two

observations is a Yes instance of 2-SARP (and hence of 2-GARP), as evidenced by the personalized quantity vectors $x_1^1 := q_1, x_2^1 := \mathbf{0}, x_1^2 := \mathbf{0}$ and $x_2^2 := q_2$. However, when assignable information is available, then instances with No answer can be built that consist of only two observations, for both 2-GARP and 2-SARP. As mentioned earlier, checking whether S satisfies GARP or SARP can be performed in time $O(n^2)$ (Varian 1982, Talla Nobibon et al. 2015).

3.2. Integer Programming Approaches Toward Testing Rationality

Over the last decades, several researchers have investigated different approaches to model and to test the rationality of collective households consumption (Bourguignon et al. 2009; Cherchye and Vermeulen 2008; Browning and Chiappori 1998; Chiappori 1988, 1992; Cherchye et al. 2010, 2007, 2012, 2011b, 2008). Some of these approaches extend unitary households revealed preference axioms to collective households (Cherchye and Vermeulen 2008; Cherchye et al. 2010, 2008, 2007, 2012, 2011b), and this is also the setting considered in our paper. The objective of the present section is not to provide an exhaustive list of papers that describe revealed preference axioms for collective households, but rather to focus on computational methods for testing collective consumption models.

Cherchye et al. (2008) rely on an integer programming formulation for testing a variant of the collective model, called *Collective Axiom of Revealed Preference*; testing this axiom was later proven to be NP-complete by Talla Nobibon and Spieksma (2010). Heuristics based on a graph-coloring approach are proposed for the same problem by Talla Nobibon et al. (2011). Deb (2011) proves that testing the “situation-dependent dictatorship” version of the collective consumption model is NP-complete, and he proposes a heuristic based on graph coloring for solving the problem.

Cherchye et al. (2011b) propose an exact testing procedure based on a MIP formulation for 2-GARP. Their formulation uses the binary decision variables y_{st}^l defined for $l = 1, 2$, and $s \neq t \in N$, with the interpretation that $y_{st}^l = 1$ if member l prefers s over t . They also introduce continuous (vectors of) decision variables x_t^l for $l = 1, 2$, and $t \in N$ corresponding with the consumption of member l in observation t . The constraints of their formulation are given by

$$x_t^1 + x_t^2 = q_t, \quad t \in N, \quad (1)$$

$$x_t^l \geq q_t^l, \quad t \in N; l = 1, 2, \quad (2)$$

$$y_{su}^l + y_{ut}^l - y_{st}^l \leq 1, \quad s \neq t \neq u \neq s \in N; l = 1, 2, \quad (3)$$

$$p'_s(x_s^l - x_t^l) - M_s y_{st}^l < 0, \quad s \neq t \in N; l = 1, 2, \quad (4)$$

$$p'_t(x_t^l - x_s^l) + p'_t q_t y_{st}^l \leq p'_t q_t, \quad s \neq t \in N; l = 1, 2, \quad (5)$$

$$y_{st}^l \in \{0, 1\}, \quad s \neq t \in N; l = 1, 2. \quad (6)$$

We refer to the formulation (1)–(6) as F1:2-GARP. The first set of constraints (1)–(2) model the split of the observed quantity vectors into feasible personalized quantity vectors. The constraints (3) enforce the transitivity of the preference relations modeled by the y_{st}^l -variables. The constraints (4) ensure that y_{st}^l takes value 1 when $p'_s x_s^l \geq p'_t x_t^l$; in these constraints, M_s is any constant such that $M_s > p'_s q_s$. The constraints (5) ensure that if $y_{st}^l = 1$, then $p'_t x_t^l \leq p'_s x_s^l$, as required by the definition of GARP. A related formulation for 2-SARP is given in the online Appendix B.

4. New Mixed-Integer Programming Formulations

The formulation F1:2-GARP given above involves $\Theta(mn)$ continuous variables, $\Theta(n^2)$ binary variables, and $\Theta(n^3)$ constraints (where the Θ -notation indicates the order of growth, up to a constant factor). For medium-sized data sets, this results in very large formulations that are hard to solve, as evidenced by our computational experiments; see Section 7. Therefore, we proceed in Sections 4.2 and 4.3 with a description of new, more compact MIP formulations for 2-GARP and for 2-SARP. To achieve this goal, however, we first need to introduce some interpretations of GARP and SARP in terms of graph theory, which will prove useful in the remainder of the paper.

4.1. Graph Interpretation of GARP and SARP

All graphs considered in this paper are *finite, directed graphs* of the form $G = (V, A)$, where V (also denoted $V(G)$) is the vertex-set of G and A is its arc-set. A graph G is *strongly connected* if, for every pair of distinct vertices i and $j \in V(G)$, there is a directed path from i to j and a directed path from j to i in G . The subgraph of G induced by $W \subseteq V$ is the graph $G(W) = (W, U)$ where U consists of all arcs whose endpoints are both in W . A *strongly connected component* (scc) of G is an induced subgraph $G(W)$ of G , which is strongly connected and such that W is maximal with respect to this property. For every directed graph G , the vertex sets of the strongly connected components of G define a (unique) partition of $V(G)$. We refer to Golombic (2004) for additional terminology and concepts.

Given the personalized consumption data set $S_l := \{(p_t, x_t^l): t \in N\}$ of member l , we build the directed graph $G^l = (N, A^l)$ as follows: Each vertex corresponds to an observation in S_l . There is an arc $s \rightarrow t \in A^l$ if and only if $s \neq t$ and $p'_s x_s^l \geq p'_t x_t^l$, that is, if $s \neq t$ and member l directly prefers s over t . We refer to G^l as the *directed graph associated* with S_l , or as the *preference graph* of member l . Let A_{scc}^l be the set of all the arcs contained in strongly connected components of G^l , so that $G_{\text{scc}}^l = (N, A_{\text{scc}}^l)$ is simply the union of the strongly connected components of G^l . The following characterizations are direct consequences of Definitions 3 and 4, respectively.

PROPOSITION 1. *The data set $S_l := \{(p_t, x_t^l): t \in N\}$ satisfies GARP if and only if every arc $s \rightarrow t \in A_{\text{scc}}^l$ satisfies $p'_s x_s^l = p'_t x_t^l$.*

PROPOSITION 2. *The data set $S_l := \{(p_t, x_t^l): t \in N\}$ satisfies SARP if and only if every arc $s \rightarrow t \in A_{\text{scc}}^l$ satisfies $x_s^l = x_t^l$.*

4.2. New MIP Formulation for 2-GARP

The formulation F1:2-GARP decomposes into two parts, with Equations (1)–(2) expressing feasibility of the personalized quantities, and Equations (3)–(6) expressing the validity of GARP for each of the personalized consumption data sets S_1 and S_2 . To obtain a new formulation for 2-GARP, we focus on the second part and we state yet another alternative characterization of GARP.

PROPOSITION 3. *The set $S_l := \{(p_t, x_t^l): t \in N\}$ satisfies GARP if and only if there exist $U_t^l \in \mathbb{R}$ for all $t \in N$ such that (i) if $p'_s x_s^l \geq p'_t x_t^l$ then $U_s^l \geq U_t^l$, and (ii) if $p'_s x_s^l > p'_t x_t^l$ then $U_s^l > U_t^l$.*

Proposition 3 is a variant of many similar results where U_t^l can be viewed as the utility level of bundle x_t^l , and the utility levels are consistent with the preference relation of member l ; see Varian (2006, 1982). For the sake of completeness and in order to prepare the ground for our next results, we provide a self-contained proof of this characterization.

PROOF. Suppose that there exist $U_t^l \in \mathbb{R}$, for all $t \in N$, satisfying (i) and (ii), and suppose that member l prefers s over t . Thus, according to Definition 2, there exists a sequence s, u, v, \dots, z, t of observations such that $p'_s x_s^l \geq p'_u x_u^l, p'_u x_u^l \geq p'_v x_v^l, \dots, p'_z x_z^l \geq p'_t x_t^l$. In view of (i), $U_s^l \geq U_u^l \geq U_v^l \geq \dots \geq U_z^l \geq U_t^l$. By (ii), this implies in turn that $p'_t x_t^l \leq p'_s x_s^l$, as required by Definition 3 of GARP.

Conversely, suppose now that S_l satisfies GARP. Consider the directed graph $G^l = (N, A^l)$ associated with S_l , and let $G_1^l, G_2^l, \dots, G_{\alpha_l}^l$ be the strongly connected components of G^l ($\alpha_l \leq n$). We can assume that the strongly connected components are indexed from 1 to α_l in *reverse topological order*, that is, in such a way that every arc of G^l goes from a component G_i^l to a component G_j^l with $i \geq j$ (see, e.g., Ahuja et al. 1993). Now, for every vertex $t \in G^l$, define $U_t^l := (i_t^l/n)$, where i_t^l is the index of the strongly connected component that contains t . Then, property (i) is satisfied by construction. On the other hand, if $p'_s x_s^l > p'_t x_t^l$, then $s \rightarrow t \in A^l$, but Proposition 1 implies that $s \rightarrow t \notin A_{\text{scc}}^l$. Hence, arc $s \rightarrow t$ has its endpoints in two distinct components, and we conclude that $U_s^l > U_t^l$. Q.E.D.

The utility levels U_t^l can explicitly be modeled as variables, thereby leading to the following MIP formulation for 2-GARP, denoted by F2:2-GARP:

$$(F2:2-GARP) \quad x_t^1 + x_t^2 = q_t, \quad t \in N, \quad (7)$$

$$x_t^l \geq q_t^l, \quad t \in N; l=1,2, \quad (8)$$

$$U_s^l - U_t^l < y_{st}^l, \quad s \neq t \in N; l=1,2, \quad (9)$$

$$y_{st}^l - 1 \leq U_s^l - U_t^l, \quad s \neq t \in N; l=1,2, \quad (10)$$

$$p'_s(x_s^l - x_t^l) - M_s y_{st}^l < 0, \quad s \neq t \in N; l=1,2, \quad (11)$$

$$p'_l(x_t^l - x_s^l) + p'_l q_{it} y_{st}^l \leq p'_l q_{it}, \quad s \neq t \in N; l=1,2, \quad (12)$$

$$y_{st}^l \in \{0, 1\}, \quad s \neq t \in N; l=1,2, \quad (13)$$

where M_s is a strict upper-bound on $p'_s q_s$. The reader should note that in this formulation, contrary to F1:2-GARP, the y_{st}^l -variables do not model the (revealed) preferences of member l , but rather reflect the order imposed by the utility levels: indeed, constraints (9)–(10) imply that, in every feasible solution, $y_{st}^l = 1$ if and only if $U_s^l \geq U_t^l$.

We have the following result.

PROPOSITION 4. *The consumption data set $S := \{(p_t, q_t; q_t^1, q_t^2): t \in N\}$ satisfies 2-GARP if and only if the domain defined by (7)–(13) is nonempty.*

PROOF. Suppose that $S := \{(p_t, q_t; q_t^1, q_t^2): t \in N\}$ satisfies 2-GARP. Then there exist $x_t^1, x_t^2 \in \mathbb{R}_+^m$ with $q_t = x_t^1 + x_t^2$ and $q_t^1 \leq x_t^1, q_t^2 \leq x_t^2$ for each observation $t \in N$ such that the consumption sets $S_l := \{(p_t, x_t^l): t \in N\}$ satisfy GARP, for $l = 1, 2$. Fix these feasible personalized quantities. For $l = 1, 2$, consider the graph $G^l = (V, A^l)$ associated with S_l and its strongly connected components $G_1^l, G_2^l, \dots, G_{\alpha_l}^l$ indexed in reverse topological order, as in the proof of Proposition 3. Recall that i_t^l is the index of the strongly connected component that contains node t . Define the point $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$, where the components of \mathbf{X} (respectively, \mathbf{Y} and \mathbf{U}) are x_t^l (respectively, y_{st}^l and U_t^l) and are specified as follows for all $s \neq t \in N$ and $l \in \{1, 2\}$:

- x_t^l is the quantity consumed by member l in observation t ;
- if $i_s^l \geq i_t^l$ then $y_{st}^l := 1$; otherwise $y_{st}^l := 0$;
- $U_t^l := (i_t^l/n)$.

We now argue that the point $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ belongs to the domain defined by (7)–(13). Indeed:

Constraints (7)–(8) and (13): These are satisfied by assumption.

Constraints (9)–(10). For $s \neq t \in N$ and $l \in \{1, 2\}$, if $i_s^l \geq i_t^l$ then $y_{st}^l = 1$ and $U_s^l \geq U_t^l$; if $i_s^l < i_t^l$ then $y_{st}^l = 0$ and $U_s^l < U_t^l$. In both cases, (9)–(10) hold since $U_t^l \in [1/n, 1]$ implies $|U_t^l - U_s^l| < 1$.

Constraints (11). For $s \neq t \in N$ and $l \in \{1, 2\}$, if $p'_s x_s^l < p'_t x_t^l$ then (11) is trivially satisfied. Otherwise, if $p'_s x_s^l \geq p'_t x_t^l$ then $s \rightarrow t \in A^l$ by construction, hence $i_s^l \geq i_t^l$ and $y_{st}^l = 1$. Therefore (11) holds.

Constraints (12). For $s \neq t \in N$ and $l \in \{1, 2\}$, if $p'_l x_s^l \geq p'_l x_t^l$ then (12) trivially holds. Otherwise, if $p'_l x_s^l < p'_l x_t^l$ then $i_s^l \neq i_t^l$ (by Proposition 1) and $y_{st}^l = 0$. Therefore (12) holds, and all constraints are satisfied.

Conversely, suppose now that $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ is in the domain defined by (7)–(13). For $l = 1, 2$, let $S_l := \{(p_t, x_t^l): t \in N\}$, where x_t^l are the components of \mathbf{X} . We now rely on Proposition 3 to prove that the personalized consumption sets $S_l := \{(p_t, x_t^l): t \in N\}$ ($l = 1, 2$) satisfy GARP.

If $p'_s x_s^l \geq p'_t x_t^l$ then constraint (11) implies that $y_{st}^l = 1$ and hence, in view of constraint (10), $U_s^l \geq U_t^l$. On the other hand, if $p'_s x_s^l < p'_t x_t^l$ then $y_{st}^l = 0$ by constraint (12), and constraint (9) implies $U_s^l < U_t^l$. Thus, properties (i) and (ii) hold in Proposition 3, and S satisfies 2-GARP. Q.E.D

Note that, as compared to F1:2-GARP, the formulation F2:2-GARP involves $2n$ additional continuous variables, but only $\Theta(n^2)$ constraints. For large values of n , this is a very significant reduction in size. One might be tempted to compare the linear programming relaxations of the two formulations F1:2-GARP and F2:2-GARP. There are, however, two issues that prevent such a comparison to be meaningful: (i) the y -variables in the two formulations have different interpretations (as mentioned above), and, perhaps more importantly, (ii) since Problem 2-GARP is a decision problem, both formulations lack an objective function.

4.3. New MIP Formulation for 2-SARP

We now extend the previous results by providing a new MIP formulation for 2-SARP. This formulation is based on Proposition 5, which is the counterpart of Proposition 3 for SARP.

PROPOSITION 5. *The set $S_l := \{(p_t, x_t^l): t \in N\}$ satisfies SARP if and only if there exist $U_t^l \in \mathbb{R}$ for $t \in N$ such that (i) $x_s^l = x_t^l$ if and only if $U_t^l = U_s^l$, and (ii) if $x_s^l \neq x_t^l$ and $p'_s x_s^l \geq p'_t x_t^l$ then $U_s^l > U_t^l$.*

PROOF. Suppose that there exist $U_t^l \in \mathbb{R}$, for all $t \in N$, satisfying (i) and (ii), and suppose that member l prefers s over t , with $x_s^l \neq x_t^l$. Thus, there exists a sequence s, u, v, \dots, z, t of observations such that $p'_s x_s^l \geq p'_u x_u^l, p'_u x_u^l \geq p'_v x_v^l, \dots, p'_z x_z^l \geq p'_t x_t^l$. Therefore, $U_s^l \geq U_u^l \geq U_v^l \geq \dots \geq U_z^l \geq U_t^l$. By (i), this implies that $U_t^l < U_s^l$, and again by (ii), we obtain $p'_t x_t^l < p'_s x_s^l$, as required by Definition 4 of SARP.

Conversely, if S_l satisfies SARP, construct U_t^l as in the proof of Proposition 3. Note that Property (ii) is satisfied as before. For (i), note that if $U_t^l = U_s^l$, then t and s are in the same strongly connected component of G^l , and hence $x_s^l = x_t^l$ by Proposition 2. On the other hand, if $x_s^l = x_t^l$, then (trivially) x_s^l is preferred over x_t^l , and x_t^l is preferred over x_s^l by member l , so that s and t are in the same component of G^l , and $U_t^l = U_s^l$. Q.E.D

Proposition 5 leads to the following MIP formulation for 2-SARP, denoted by F2:2-SARP:

$$(F2:2-SARP) \quad x_t^1 + x_t^2 = q_t, \quad t \in N, \quad (14)$$

$$x_t^l \geq q_t^l, \quad t \in N; l=1,2. \quad (15)$$

$$U_s^l - U_t^l < y_{st}^l, \quad s \neq t \in N; l=1,2, \quad (16)$$

$$y_{st}^l - 1 \leq U_s^l - U_t^l, \quad s \neq t \in N; l=1,2, \quad (17)$$

$$p'_s(x_s^l - x_t^l) - M_s y_{st}^l < 0, \quad s \neq t \in N; l=1,2, \quad (18)$$

$$\frac{1}{q_{s,i}}(x_{s,i}^l - x_{t,i}^l) \leq 2 - y_{st}^l - y_{ts}^l,$$

$$s \neq t \in N; i = 1, \dots, m; l = 1, 2, \quad (19)$$

$$y_{st}^l \in \{0, 1\}, \quad s \neq t \in N; l = 1, 2, \quad (20)$$

where M_s is a strict upper bound on $p'_s q_s$.

PROPOSITION 6. *The consumption data set $S := \{(p_t, q_t; q_t^1, q_t^2): t \in N\}$ satisfies 2-SARP if and only if the domain defined by (14)–(20) is nonempty.*

PROOF. The proof is a slight modification of that of Proposition 4. If S satisfies 2-SARP, then define $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ as in the proof of Proposition 4. Since SARP implies GARP, we already know that constraints (14)–(18) and (20) are satisfied. For (19), note that if $x_s^l \neq x_t^l$ then necessarily $i_s^l \neq i_t^l$ by Proposition 2. Hence the construction of \mathbf{Y} entails $y_{st}^l + y_{ts}^l \leq 1$, so that (19) is satisfied.

Conversely, if $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ is a solution of (14)–(20), then again, \mathbf{U} satisfies condition (ii) of Proposition 5. For condition (i), assume that $U_s^l = U_t^l$. Then, because of constraints (16), $y_{st}^l = y_{ts}^l = 1$, and constraints (19) (applied to the pairs (s, t) and (t, s)) imply $x_s^l = x_t^l$. The converse implication follows from constraints (17) and (18). Q.E.D

The formulation F2:2-SARP involves the same variables as F2:2-GARP, and $\Theta(mn^2)$ constraints. In practical applications m is typically much smaller than n (see Section 7), and therefore the new formulation is more compact than the formulation given in the online Appendix B.

4.4. Additional Comments About the MIP Formulations

For computational purposes, the formulations F2:2-GARP and F2:2-SARP can be improved in a number of ways. First, it immediately follows from the proofs of Propositions 4 and 6 that all variables U_s^l can be restricted to lie between 0 and 1, for all $s \neq t \in N$ and $l = 1, 2$. Moreover, a closer examination of these proofs reveals that the constraints (9) and (16), which involve strict inequalities, can be replaced by the stronger constraints:

$$U_s^l - U_t^l \leq y_{st}^l - \frac{1}{n}, \quad s \neq t \in N; l = 1, 2. \quad (21)$$

In the sequel, whenever we refer to the MIP formulations F2:2-GARP and F2:2-SARP, we implicitly assume that $U_s^l \in [0, 1]$ and that (9) and (16) have been replaced by (21).

After these modifications, both MIP formulations still involve the strict inequalities (11) (or (18)), which are not desirable from the numerical point of view. However, if the domain of F2:2-GARP is nonempty, then there exists a small positive ε such that the domain remains nonempty when we replace the constraints (11) by

$$p'_s(x_s^l - x_t^l) - M_s y_{st}^l \leq -\varepsilon, \quad s \neq t \in N; l = 1, 2. \quad (22)$$

Indeed, suppose that $(\mathbf{X}, \mathbf{Y}, \mathbf{U})$ satisfies all constraints (7)–(13). Fixing the values of y_{st}^l (for all $s \neq t \in N$ and $l = 1, 2$) in the constraints yields a feasible system of linear inequalities. From linear programming theory (Chvátal 1983), we know that there exists a small

$\varepsilon > 0$ such that this system of inequalities remains feasible if we replace the right-hand side of (11) by $-\varepsilon$. Since the number of possible assignments of binary values to \mathbf{Y} is finite, it follows that there is a smallest value of ε that can be used without affecting the feasibility of F2:2-GARP. The same reasoning applies to the formulation F2:2-SARP. Note that M_s should be at least $p'_s q_s + \varepsilon$ for the previous reasoning to be valid.

Finally, we end this section by describing a set of cuts (that is, valid inequalities) that can be used to strengthen the formulations F2:2-GARP and F2:2-SARP. Note that, in the proof of Proposition 4, at least one of y_{st}^l and y_{ts}^l takes value 1, for all $s \neq t \in N$. This condition actually holds for every feasible solution, as can be deduced by taking the sum of the constraints (9), respectively, associated with the pairs (s, t) and (t, s) . As a consequence, the following inequalities are valid and can safely be added to each of the formulations F2:2-GARP and F2:2-SARP without affecting their feasibility:

$$y_{st}^l + y_{ts}^l \geq 1, \quad s < t \in N; l = 1, 2. \quad (23)$$

The effect of these cuts on the solvability of the MIP models is investigated in Section 7.

4.5. Extension to $L \geq 2$ Household Members

It is not difficult to generalize both F2:2-GARP and F2:2-SARP to a setting where the household consists of an arbitrary (yet given) number of members, say L . To formulate the corresponding L -GARP and L -SARP problems, it is enough to replace constraints (7) and (14) by

$$\sum_{l=1}^L x_t^l = q_t \quad t \in N \quad (24)$$

and to extend all constraints of F2:2-GARP or F2:2-SARP indexed by l to $l = 1, \dots, L$.

The proofs of Proposition 4 and 6 immediately establish that the formulations obtained in this way are valid for L -GARP and L -SARP.

We also remark here that typical applications of collective consumption models include only two decision makers per household (see, for example, the original contributions of Chiappori 1988, 1992, and Apps and Rees 1988). For households with children, the associated expenditures are usually modeled as public consumption (see, for example, Blundell et al. 2005). Notable exceptions are Bargain and Donni (2012) and Dunbar et al. (2013), who indeed treat children as separate decision makers in the household.

In our computational experiments, we only work with $L = 2$. It would be interesting to see how the MIP models behave in case of $L \geq 3$. We leave this for further research.

5. Global Optimization Formulations

The MIP formulations presented in previous sections turn out to be hard to solve for large instances of 2-SARP and

2-GARP. Therefore, we have also developed heuristic algorithms that rely on global optimization formulations to be presented in Section 5.1. In Section 5.2, we describe a graph-based representation of solutions and we discuss some of their properties.

5.1. Global Optimization Formulations for 2-GARP and 2-SARP

In the sequel, for ease of exposition, we often use the variables $x_s := x_s^1$, and we implicitly set $x_s^2 := q_s - x_s^1$ for all $s \in N$. Consider the box $\mathbf{D} := \{(x_1, x_2, \dots, x_n) : q_s^1 \leq x_s \leq q_s - q_s^2, \forall s \in N\}$ corresponding to the set of feasible personalized quantities, and denote by $\mathbf{X} := (x_1, x_2, \dots, x_n) \in \mathbf{D}$ an arbitrary point in this box. Each such point \mathbf{X} can be associated with personalized data sets $S_l := \{(p_t, x_t^l) : t \in N\}$ and with their associated preference graphs $G^l = (N, A^l)$, for $l = 1, 2$. Let $\hat{A}_{\text{sec}}^l \subseteq A_{\text{sec}}^l$ be the subset of A_{sec}^l containing all arcs $s \rightarrow t \in A_{\text{sec}}^l$ such that $p'_s x_s^l \neq p'_t x_t^l$. Similarly, let $\tilde{A}_{\text{sec}}^l \subseteq A_{\text{sec}}^l$ contain all arcs $s \rightarrow t \in A_{\text{sec}}^l$ with $x_s^l \neq x_t^l$. Note that we have the inclusion $\hat{A}_{\text{sec}}^l \subseteq \tilde{A}_{\text{sec}}^l$. Propositions 1 and 2 are equivalently reformulated as follows.

PROPOSITION 7. *The data set $S_l := \{(p_t, x_t^l) : t \in N\}$ satisfies GARP if and only if \hat{A}_{sec}^l is empty.*

PROPOSITION 8. *The data set $S_l := \{(p_t, x_t^l) : t \in N\}$ satisfies SARP if and only if \tilde{A}_{sec}^l is empty.*

Let us now introduce the following integer-valued functions on the domain \mathbf{D} :

$$f_g(\mathbf{X}) = f_g(x_1, x_2, \dots, x_n) := |\hat{A}_{\text{sec}}^1| + |\hat{A}_{\text{sec}}^2|, \quad (25)$$

$$f_s(\mathbf{X}) = f_s(x_1, x_2, \dots, x_n) := |\tilde{A}_{\text{sec}}^1| + |\tilde{A}_{\text{sec}}^2|, \quad (26)$$

and consider the unconstrained optimization problems

$$\begin{aligned} \text{(G:2-GARP)} \quad & \min f_g(\mathbf{X}) \\ & \text{s.t. } \mathbf{X} \in \mathbf{D}, \end{aligned}$$

and

$$\begin{aligned} \text{(G:2-SARP)} \quad & \min f_s(\mathbf{X}) \\ & \text{s.t. } \mathbf{X} \in \mathbf{D}. \end{aligned}$$

PROPOSITION 9. *The consumption data set $S := \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$ is a feasible instance of 2-GARP (respectively, 2-SARP) if and only if the optimal value of problem G:2-GARP (respectively, G:2-SARP) is equal to 0.*

PROOF. This is immediate in view of Propositions 7 and 8. Q.E.D

For each $\mathbf{X} \in \mathbf{D}$, the functions $f_s(\mathbf{X})$ and $f_g(\mathbf{X})$ can be evaluated in polynomial time. Note, however, that they are neither continuous nor convex, so that G:2-GARP and G:2-SARP must be viewed as *global optimization problems*.

When comparing the formulation G:2-GARP with F2:2-GARP, we observe that the constraints (7)–(8) expressing

feasibility of the personalized quantities are enforced by $\mathbf{X} \in \mathbf{D}$, whereas the constraints (9)–(13) expressing the validity of GARP for each of the personalized consumption data sets S_1 and S_2 are transferred to the objective function of G:2-GARP. Similar observations apply to G:2-SARP and F2:2-SARP. Many other objective functions could be substituted for $f_s(\mathbf{X})$ and for $f_g(\mathbf{X})$ in G:2-GARP and in G:2-SARP. The functions (25) and (26) have been chosen because they are well suited for the algorithmic developments described in subsequent sections.

5.2. Graphs Associated with a Solution

We have seen that every feasible solution $\mathbf{X} \in \mathbf{D}$ can be associated with a unique pair of directed graphs (G^1, G^2) . But the converse does not hold: Clearly, certain pairs of directed graphs on N are not associated with any point in \mathbf{D} . Moreover, two points \mathbf{X} and \mathbf{X}' in \mathbf{D} may give rise to the same pair of graphs (G^1, G^2) . When this is the case, we say that \mathbf{X} and \mathbf{X}' are *equivalent*: indeed, then, Propositions 7–8 imply that knowledge of (G^1, G^2) suffices to determine whether the personalized data sets associated with each of \mathbf{X}, \mathbf{X}' satisfy GARP or SARP, and we actually have $f_g(\mathbf{X}) = f_g(\mathbf{X}')$ and $f_s(\mathbf{X}) = f_s(\mathbf{X}')$.

Thus, the pair of graphs (G^1, G^2) associated with $\mathbf{X} \in \mathbf{D}$ can be viewed as providing an incomplete, but sufficient, representation of \mathbf{X} . The main advantage of this representation is its *combinatorial* structure: namely, the number of pairs (G^1, G^2) is finite, as opposed to the number of points in \mathbf{D} . Therefore, the representation based on graph pairs better lends itself to the application of local search approaches.

To understand what solutions are associated with a given pair of graphs, let us consider the following decision problem:

Problem PC(G^1, G^2)

Instance: A data set $S = \{(p_t, q_t; q_t^1, q_t^2) : t \in N\}$ and two directed graphs $G^1 = (N, A^1)$ and $G^2 = (N, A^2)$.

Question: Does there exist a point $\mathbf{X} \in \mathbf{D}$ such that the pair of directed graphs associated with \mathbf{X} is exactly (G^1, G^2) ?

We argue now that problem PC(G^1, G^2) can be solved in polynomial time. For an (unknown) point $\mathbf{X} \in \mathbf{D}$ and for each pair $s, t \in N, s \neq t$, of distinct observations, we are going to set up inequalities of the following type:

$$lhs[s, t] \leq p'_s x_s - p'_t x_t \leq rhs[s, t]. \quad (27)$$

The values of the left-hand side $lhs[s, t]$ and of the right-hand side $rhs[s, t]$ in (27) are set according to Table 1, where ε is a very small and positive number, and NA means *nonapplicable*.

We denote by $\mathcal{P}(G^1, G^2)$ the system of $n(n - 1)$ constraints of type (27) constructed in this way, together with all the constraints enforcing assignable restrictions: $q_s^1 \leq x_s \leq q_s - q_s^2$, for all $s \in N$. The system $\mathcal{P}(G^1, G^2)$ involves $\Theta(mn)$ continuous variables and $\Theta(n^2)$ constraints. We have the following straightforward result.

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Table 1. Values of $lhs[s, t]$ and $rhs[s, t]$ when G^1 and G^2 are given.

Four cases	$lhs[s, t]$	$rhs[s, t]$
$s \rightarrow t \in A^1$ and $s \rightarrow t \in A^2$	0	$p'_s q_s - p'_s q_t$
$s \rightarrow t \in A^1$ and $s \rightarrow t \notin A^2$	$\max\{0, p'_s q_s - p'_s q_t + \varepsilon\}$	NA
$s \rightarrow t \notin A^1$ and $s \rightarrow t \in A^2$	NA	$\min\{-\varepsilon, p'_s q_s - p'_s q_t\}$
$s \rightarrow t \notin A^1$ and $s \rightarrow t \notin A^2$	$p'_s q_s - p'_s q_t + \varepsilon$	$-\varepsilon$

PROPOSITION 10. For ε sufficiently small, the solutions to the system $\mathcal{P}(G^1, G^2)$ are exactly the points $\mathbf{X} \in \mathbf{D}$ such that the pair of graphs (G^1, G^2) is associated with \mathbf{X} . In particular, the system $\mathcal{P}(G^1, G^2)$ has a solution if and only if the problem $\text{PC}(G^1, G^2)$ has a Yes answer, and these conditions can be tested in polynomial time.

PROOF. For all $\mathbf{X} \in \mathbf{D}$, the inequalities (27) simply express the conditions under which an arc is either present or absent in each of the graphs G^1, G^2 . Since the system $\mathcal{P}(G^1, G^2)$ is a linear program, we infer that problem $\text{PC}(G^1, G^2)$ can be decided in polynomial time (Khachiyan 1979, Karmarkar 1984). Q.E.D

The above analysis requires only minor adaptations to work when G^1 and G^2 represent partial preferences. More precisely, we have the following:

PROPOSITION 11. If E^1, F^1, E^2, F^2 are subsets of $N \times N$ such that $E^1 \cap F^1 = E^2 \cap F^2 = \emptyset$, one can test in polynomial time whether there exists a point $\mathbf{X} \in \mathbf{D}$ such that the graphs $G^1 = (N, A^1)$ and $G^2 = (N, A^2)$ associated with \mathbf{X} satisfy $E^l \subseteq A^l$ and $A^l \cap F^l = \emptyset$ for $l = 1, 2$.

PROOF. In the statement, E^l stands for the set of direct preferences that must be present in G^l , whereas F^l stands for the set of preferences that must be absent from G^l , for each $l = 1, 2$. The conclusion follows, since the conditions on \mathbf{X} can be tested by imposing the inequalities (27) for all pairs $(s, t) \in E^1 \cup F^1 \cup E^2 \cup F^2$, and by adapting the rows of Table 1 accordingly. Q.E.D

Let us shortly address the question to what extent the results in this section can be generalized to L members in the household. Clearly, Propositions 7–9 directly generalize. Of course, now we need to understand what solutions are associated with a given set of L graphs, i.e., we now face the following decision question:

Problem $\text{PC}(G^1, G^2, \dots, G^L)$

Instance: A data set $S = \{(p_t, q_t; q_t^1, q_t^2, \dots, q_t^L) : t \in N\}$ and L directed graphs $G^1 = (N, A^1), G^2 = (N, A^2), \dots, G^L = (N, A^L)$.

Question: Does there exist a point $\mathbf{X} \in \mathbf{D}$ such that the L directed graphs associated with \mathbf{X} are exactly G^1, G^2, \dots, G^L ?

Now, the presence of an arc in each G^l , as well as the absence of an arc in each G^l ($1 \leq l \leq L$) gives rise to a (strict) inequality. Also, we have the equalities arising from (24). The resulting constraints are all linear, and there are $O(mn)$ variables with $O(Ln^2)$ constraints, so the resulting formulation can still be solved in polynomial time. Hence, a generalized version of Proposition 10 remains true. As a result of these observations, the simulated annealing algorithm described in the next section could be extended in a rather straightforward way to the case of L household members, $L > 2$. Implementing this extended algorithm and testing its efficiency on appropriate data sets, however, would require extensive computational work. Also, as we already observed in Section 4.5, such extensions do not appear to be well justified by the economic applications. Therefore, we restrict our attention from now on to the case $L = 2$.

6. Simulated Annealing Algorithm

We present a simulated annealing (SA) algorithm for solving G:2-GARP and G:2-SARP. Similar algorithms have been successfully applied to global optimization problems where the objective function is neither continuous nor convex; see, e.g., Dekkers and Aarts (1991), Goffe et al. (1994), Locatelli (2000).

The SA algorithm implemented in this paper follows the scheme described by Dekkers and Aarts (1991), and requires the following parameters: A nonnegative real number c , called *control parameter*, which is used to determine whether a solution with an objective value worse than that of the current solution is accepted as the next current solution or not. A positive integer κ representing the length of each *Markov chain* (sequence of trials), and the function $r[0, 1)$ that generates uniformly a random value in $[0, 1)$. Finally, *stop* is a Boolean function used to halt the algorithm. The pseudocode of the SA algorithm is depicted in Algorithm 1, where $f(\mathbf{X})$ can be replaced either by $f_g(\mathbf{X})$ or by $f_s(\mathbf{X})$. Concretely, the algorithm starts from an initial solution \mathbf{X} and a given value of c . The function *stop* is initialized to “false.” While *stop* remains false, the algorithm generates at each iteration a sequence of κ trial solutions. Each solution can be set as the new current solution, depending on whether its objective value is smaller than that of the current solution or, if not, based on the value of the control parameter c . This procedure is repeated until either a solution with objective value zero is found, or the function *stop* takes the value “true.” Upon termination, the algorithm outputs either “Yes” or “undecided.” Observe that, since SA is not an exact optimization algorithm, it may sometimes return “undecided” when running on “Yes” instances. This case will be examined in Section 7.

Algorithm 1 (Simulated annealing)

- 1: initialize $\mathbf{X} := \mathbf{X}_0$, $c := c_0$, and *stop* := false
- 2: **while** *stop* = false **do**

- 3: **for** $i := 1$ to κ **do**
- 4: generate \mathbf{X}' either as a neighbor of \mathbf{X} or as a random solution
- 5: **if** $f(\mathbf{X}') = 0$ **then** $\mathbf{X} := \mathbf{X}'$, and go to 9
- 6: **else if** $f(\mathbf{X}') < f(\mathbf{X})$ **then** $\mathbf{X} := \mathbf{X}'$
- 7: **else if** $\exp\left(\frac{f(\mathbf{X}) - f(\mathbf{X}')}{c}\right) > r[0, 1)$ **then** $\mathbf{X} := \mathbf{X}'$
- 8: decrease the value of c and update *stop*
- 9: **if** $f(\mathbf{X}) = 0$ **then** return “Yes”
- 10: **else** return “Undecided”

6.1. Preprocessing

In this section, we present a few conditions, which, when satisfied, allow us to directly conclude that a given instance of 2-GARP or 2-SARP is either a Yes-instance or a No-instance. Since 2-SARP is stronger than 2-GARP, we focus on 2-SARP when identifying Yes-instances and on 2-GARP for No-instances. We separately consider the cases with and without assignable information.

Let first S be a consumption data set without assignable information (i.e., $q_t^1 = q_t^2 := \mathbf{0}$, for all $t \in N$) and let $G = (N, A)$ be the directed graph associated with S : each observation in S defines a vertex, and there is an arc $s \rightarrow t \in A$ if and only if $p'_s q_s \geq p'_t q_t$.

LEMMA 1. *If the vertices of G can be partitioned into two subsets such that each subset induces an acyclic subgraph, then S satisfies 2-SARP, and hence 2-GARP.*

PROOF. Suppose that N can be partitioned into two subsets N_1 and N_2 such that each subset induces an acyclic subgraph. We define personalized quantity vectors as follows. For all $s \in N_1$, we set $x_s^1 := q_s$ and $x_s^2 := \mathbf{0}$, and for all $s \in N_2$, we set $x_s^1 := \mathbf{0}$ and $x_s^2 := q_s$. It is not difficult to see that the resulting personalized consumption data sets S_l , for $l = 1, 2$, satisfy SARP, and hence GARP. Q.E.D

The setting exploited in Lemma 1 is known in economics as *situation-dependent dictatorship* because, for each observation, the complete bundle is consumed by one member of the household (Cherchye et al. 2007, Deb 2011). Note that testing the conditions stated in Lemma 1 requires testing whether graph G can be partitioned into two acyclic subgraphs, and the latter problem is known to be NP-complete (Talla Nobibon et al. 2012). We next present two properties that can be checked in polynomial time.

Let us define the vectors $\tilde{q}_t^1 := q_t - q_t^2$ and $\tilde{q}_t^2 := q_t - q_t^1$, for $t \in N$. Consider the directed graphs $\tilde{H}^1 = (N, \tilde{A}_1)$ and $\tilde{H}^2 = (N, \tilde{A}_2)$ where there is an arc $s \rightarrow t \in \tilde{A}_1$ (respectively, $s \rightarrow t \in \tilde{A}_2$) if and only if $p'_s \tilde{q}_s^1 \geq p'_t \tilde{q}_t^1$ (respectively, $p'_s \tilde{q}_s^2 \geq p'_t \tilde{q}_t^2$).

LEMMA 2. *If \tilde{H}^1 and \tilde{H}^2 are acyclic, then S satisfies 2-SARP, and hence 2-GARP.*

PROOF. Note that for any pair of feasible personalized consumption sets $S_l := \{(p_t, x_t^l) : t \in N\}$, $l = 1, 2$, member l consumes at least q_t^l and at most \tilde{q}_t^l in each observation

$t: q_t^l \leq x_t^l \leq \tilde{q}_t^l$. Therefore, the directed graph $G^l = (V, A^l)$ associated with S_l is a subgraph of \tilde{H}^l , and when \tilde{H}^l is acyclic, its subgraph G^l is also acyclic. This implies that S_l ($l = 1, 2$) satisfies SARP, and hence GARP. Q.E.D

The next result identifies a condition that leads to a No answer. We consider the directed graphs $\tilde{H}^1 = (N, \tilde{A}_1)$ and $\tilde{H}^2 = (N, \tilde{A}_2)$ where there is an arc $s \rightarrow t \in \tilde{A}_1$ (respectively, $s \rightarrow t \in \tilde{A}_2$) if and only if $p'_s q_s^1 > p'_t \tilde{q}_t^1$ (respectively, $p'_s q_s^2 > p'_t \tilde{q}_t^2$).

LEMMA 3. *If either \tilde{H}^1 or \tilde{H}^2 contains a cycle, then S does not satisfy 2-GARP, and hence S does not satisfy 2-SARP.*

PROOF. The proof follows a similar reasoning as that of Lemma 2. Q.E.D

All conditions described by Lemmas 1–3 can be verified in a preprocessing step before the core of the SA algorithm is executed, as explained in Section 7.2.

6.2. Initialization

We first try to produce an initial solution by successively considering the following procedures $P1$, $P2$, and $P3$. Each of these procedures proceeds by building a pair of directed graphs (G^1, G^2) that represent either the full preferences or the partial preferences of the two household members.

$P1$: Let $\pi: N \rightarrow \{1, 2, \dots, n\}$ be an arbitrary (random) permutation of the set of observations. We build (G^1, G^2) as follows: for every pair $s \neq t \in N$, we set $s \rightarrow t \in A^1$ if and only if $\pi(s) < \pi(t)$, and $s \rightarrow t \in A^2$ otherwise. The resulting graphs G^1 and G^2 are tentatively taken to represent the complete preference graphs of members 1 and 2. Note that they are acyclic. Therefore, if $\mathcal{P}(G^1, G^2)$ is nonempty then the SA algorithm stops and returns Yes: the instance satisfies 2-SARP.

After having tested a number of permutations π , if we have not been able to conclude that the data set satisfies 2-SARP, then we continue with procedure $P2$ hereunder, where $G = (N, A)$ is the directed graph associated with S .

$P2$: For every pair $s \neq t \in N$, if $s < t$ and $s \rightarrow t \in A$, then we impose $s \rightarrow t \in A^1$ and $s \rightarrow t \notin A^2$. Otherwise, if $s > t$ and $s \rightarrow t \in A$, then we impose $s \rightarrow t \notin A^1$ and $s \rightarrow t \in A^2$. Note that we do not formulate any condition when $s \rightarrow t \notin A$, so that the preference graphs are only partially determined by these conditions. Proposition 11 can be used to find a point $\mathbf{X}_0 \in \mathbf{D}$ such that the graphs associated with \mathbf{X}_0 satisfy the imposed conditions, if there is one. Any such point provides an initial solution for SA.

When $P2$ fails to provide an initial solution, we turn to the following procedure.

$P3$: For every pair $s \neq t \in N$, if $s > t$ and $s \rightarrow t \in A$, then we impose $s \rightarrow t \notin A^1$ and $s \rightarrow t \in A^2$. Once again, any point \mathbf{X}_0 satisfying these conditions can be used to initialize SA.

Finally, when all three procedures $P1$, $P2$, and $P3$ fail, we simply choose $\mathbf{X}_0 := \text{random}()$, where $\text{random}()$ is a function that randomly generates a point in \mathbf{D} . In our

numerical experiments, this function is implemented as follows: for each observation s and for each good i , we set $x_{s,i} := \alpha q_{s,i}^1 + (1 - \alpha)(q_{s,i} - q_{s,i}^2)$, where α is a random value in $[0, 1]$.

This two-step procedure was adopted after some preliminary experiments involving a comparison with a purely random initial solution, or with a variant of procedures $P2$, $P3$ where the arc-sets A^1 and A^2 were extended to their transitive closures. These alternative implementations, however, did not produce better results than the one described.

We adopt the procedure proposed by Dekkers and Aarts (1991) to initialize the control parameter c : namely, given \mathbf{X}_0 , the initial value $c = c_0$ is chosen large enough so that almost all randomly generated transitions in \mathbf{D} are accepted by SA.

6.3. Neighborhood of a Feasible Solution and Generation of New Solutions

To describe the neighborhood $\mathcal{N}(\mathbf{X})$ of a feasible solution $\mathbf{X} \in \mathbf{D}$, let (G^1, G^2) be the pair of preference graphs associated with \mathbf{X} as in Section 5.1. We want the neighbors $\mathbf{X}' \in \mathcal{N}(\mathbf{X})$ to be associated with a pair of preference graphs that is different from (G^1, G^2) , since otherwise \mathbf{X} and \mathbf{X}' could be viewed as equivalent, as explained in Section 5.2. Accordingly, we define $\mathcal{N}(\mathbf{X})$ as follows. Let A_{scc}^1 and A_{scc}^2 denote the arc-sets of the strongly connected components of G^1 and G^2 , respectively. For each arc $s \rightarrow t \in A_{scc}^1 \cup A_{scc}^2$, we define a subset $\mathcal{N}_{st}(\mathbf{X})$ of the neighborhood of \mathbf{X} ; namely, $\mathcal{N}_{st}(\mathbf{X})$ contains all solutions $\mathbf{X}' \in \mathbf{D}$ such that the associated pair of directed graphs (H^1, H^2) , $H^1 = (N, E^1)$, $H^2 = (N, E^2)$, satisfies the following conditions:

- if $s \rightarrow t \in A^1$, then $A^1 \setminus \{s \rightarrow t\} \subseteq E^1$, $s \rightarrow t \notin E^1$, and $A^2 \subseteq E^2$;
- else, $A^1 \subseteq E^1$, $A^2 \setminus \{s \rightarrow t\} \subseteq E^2$, and $s \rightarrow t \notin E^2$.

Finally, the neighborhood of \mathbf{X} is expressed as $\mathcal{N}(\mathbf{X}) := \bigcup_{s \rightarrow t \in A_{scc}^1 \cup A_{scc}^2} \mathcal{N}_{st}(\mathbf{X})$.

The rationale behind this definition is to try to remove arcs $s \rightarrow t$ from $A_{scc}^1 \cup A_{scc}^2$, so as to decrease the value of the objective function (25) or (26). The condition of the form $s \rightarrow t \notin E^l$, where $l \in \{1, 2\}$, ensures that $(H^1, H^2) \neq (G^1, G^2)$. (More complex neighborhoods could be designed by considering simultaneously several arcs in $A_{scc}^1 \cup A_{scc}^2$, but this option has not been investigated.)

Note that $\mathcal{N}(\mathbf{X})$ does not contain \mathbf{X} nor any solution equivalent to \mathbf{X} . Therefore, there is no guarantee that $\mathcal{N}(\mathbf{X})$ is nonempty. Furthermore, $\mathcal{N}(\mathbf{X})$ is described implicitly because its definition relies on the preference graphs, rather than points in \mathbf{D} .

Given a point $\mathbf{X} \in \mathbf{D}$ and an arc $s \rightarrow t \in A_{scc}^1 \cup A_{scc}^2$, we can rely again on Proposition 11 to find a point \mathbf{X}'_{st} in $\mathcal{N}_{st}(\mathbf{X})$ or to prove that $\mathcal{N}_{st}(\mathbf{X})$ is empty. In our implementation, when we need to find a point in $\mathcal{N}(\mathbf{X})$, we run this procedure over successive choices of $s \rightarrow t \in A_{scc}^1 \cup A_{scc}^2$, as prescribed by the local search procedure $LS(\mathbf{X})$ (see Algorithm 2). $LS(\mathbf{X})$ searches within the neighborhood $\mathcal{N}(\mathbf{X})$ for a solution with a better value than \mathbf{X} . If it cannot find

such an improving solution, then the algorithm returns the best neighbor found when $\mathcal{N}(\mathbf{X})$ is nonempty, and it returns a random solution otherwise.

Algorithm 2 ($LS(\mathbf{X})$)

- 1: **for all** $s \rightarrow t \in A_{scc}^1 \cup A_{scc}^2$ **do**
- 2: **if** $\mathcal{N}_{st}(\mathbf{X}) \neq \emptyset$ **then** find $\mathbf{X}'_{st} \in \mathcal{N}_{st}(\mathbf{X})$
- 3: **if** $f(\mathbf{X}'_{st}) < f(\mathbf{X})$ **then** set $\mathbf{X}' := \mathbf{X}'_{st}$ and go to 6
- 4: **if** $\mathcal{N}(\mathbf{X}) = \emptyset$ **then** generate $\mathbf{X}' := random()$ randomly and go to 6
- 5: **else** let \mathbf{X}'_{st} be the best solution found in $\mathcal{N}(\mathbf{X})$ and set $\mathbf{X}' := \mathbf{X}'_{st}$
- 6: **return** \mathbf{X}'

Finally, Step 3 of the simulated algorithm SA can be implemented as follows: given a solution \mathbf{X} , we generate a new solution \mathbf{X}' using either the function $random()$ or the local search procedure. The choice between these two approaches is governed by the following rule:

$$\mathbf{X}' := \begin{cases} LS(\mathbf{X}) & \text{with probability } 1 - \gamma, \\ random() & \text{otherwise,} \end{cases} \quad (28)$$

where γ is a fixed number in the interval $[0, 1]$. Here again, we see that the new solution \mathbf{X}' does not need to improve the current one and can even be random. This allows SA to diversify the search and to avoid the traps of local optima.

6.4. Control Parameter and Stopping Criterion

After each iteration of the SA algorithm (characterized by the generation of a Markov chain consisting of κ trial solutions), the new value for the control parameter, say c' , is computed based on the current value c using the following expression (Dekkers and Aarts 1991):

$$c' := c \left(1 + \frac{c \log(1 + \delta)}{3\sigma(c)} \right), \quad (29)$$

where $\sigma(c)$ is the standard deviation of the objective value of the κ solutions in the Markov chain at c , and δ is a given parameter.

We stop the SA algorithm when either we have found a solution with objective value 0 (and return “Yes”), or the time limit is reached, or the value of c is smaller than a predefined threshold (and return “Undecided”).

7. Computational Experiments

We present the real-world data used for our experiments in Section 7.1, we describe some issues related to the implementation of the algorithms in Section 7.2, and we subsequently discuss computational results in Section 7.3.

7.1. Data

The algorithms described in this paper are used for testing the rationality of the behavior of Dutch households. More precisely, we use the data sample presented in Cherchye

and Vermeulen (2008). This sample is based on 11 waves of the DNB Household Survey (formerly known as the CentER Savings Survey), drawn from 1995 to 2005. Each household consists of a couple, and each member of each couple is employed, is aged between 25 and 55, and has no child. The goods are aggregated into three groups: the first group contains goods that are shared by the two members of the household, the second one contains leisure female goods, and the third one contains leisure male goods. Natural assignable information supposes that goods in the second group are consumed exclusively by one member, and goods in the third group are consumed exclusively by the other member. For ease of exposition, in the rest of this section we simply assume that we have three goods. We refer to Cherchye and Vermeulen (2008) for additional details about the data.

The complete consumption data set contains more than 500 observations. From this large data set, we create smaller ones as follows. For each value of n ($n \geq 10$), we generate 20 instances, where each instance contains n randomly drawn distinct observations. We organize all the instances into four sets. The first one, denoted Data I, contains all the data sets generated for different values of n , where assignable information is ignored. This corresponds with the setting where each member in the household can consume any good. The second set of instances, Data II, contains the same instances as Data I, but with assignable information specifying that the second good is consumed exclusively by one member, whereas the third good is consumed exclusively by the other member. As an aside, let us mention here that assignable consumption information may impact the testable implications of transitivity of the individual preference relations; see Smeulders et al. (2015) for a detailed discussion.

We create two additional collections of data sets by adding proportional assignable information $q_{i,1}^1$ on nonassignable good(s) for the instances in Data I and Data II. We consider seven different pairs of proportions $(q_{i,1}^1, q_{i,1}^2)$: (20, 70), (25, 70), (40, 40), (40, 45), (45, 40), (70, 25), and (70, 20). In the first setting, for instance, the assumption is that member 1 consumes at least 20% of the first good ($q_{i,1}^1 = 0.20$), whereas member 2 consumes at least 70% of the first good ($q_{i,1}^2 = 0.70$). The third (respectively, fourth) set Data III (respectively, Data IV) contains instances from Data I (respectively, Data II) with at most $n = 100$ observations. Each instance in Data I (respectively, Data II) gives rise to seven instances in Data III (respectively, Data IV), corresponding with the seven pairs listed above. All the instances described in this section can be found at <http://users.aims.ac.za/~tal/programmingcodes.html>.

7.2. Implementation Issues

In this section, we provide some details related to the implementation of the different solution approaches. The MIP formulations F1:2-GARP, F1:2-SARP, F2:2-GARP, and F2:2-SARP are solved using CPLEX 12.4. More precisely,

each formulation is turned into an optimization problem by adding an objective function to it. We experimented with the following four alternative objective functions, namely, (*obj1*) maximize the consumption of member 1 (sum of the variables x_i^1); (*obj2*) maximize the “number of preferences” of member 1 (sum of the variables y_{st}^1); (*obj3*) maximize the total consumption difference between the two members (sum of the differences $x_i^1 - x_i^2$); and (*obj4*) minimize a constant. We halt the solver after the first feasible solution is found, and a time limit of 30 minutes is imposed on the computation time. In these MIP formulations, we set $\varepsilon := 10^{-6}$, which is the tolerance value for CPLEX, and we use $M_s := p_s q_s + 1$ in constraints (22). For F2:2-GARP and F2:2-SARP, we also consider alternative implementations that include the cuts describe by Equation (23).

For the SA algorithm, a preprocessing phase consists in checking the conditions of Lemma 1–3. We use the backtracking algorithm proposed by Talla Nobibon et al. (2012) to investigate whether the vertices of a directed graph can be partitioned into two acyclic subgraphs (with a time limit of two minutes), and we use the topological ordering algorithm (Ahuja et al. 1993) to check whether a directed graph is acyclic. The value of the parameters of the SA algorithm are chosen as proposed (and used) by Dekkers and Aarts (1991): $\delta := 0.1$, $\kappa := 10$ and $\gamma := 0.75$. The Boolean indicator *stop* takes the value “true” when either $c < 10^{-4}$ or when the running time of the algorithm exceeds a limit of 30 minutes. Furthermore, *stop* is also set to “true” after 20 iterations of SA without improvement of the objective function. This value was chosen after some preliminary experiments.

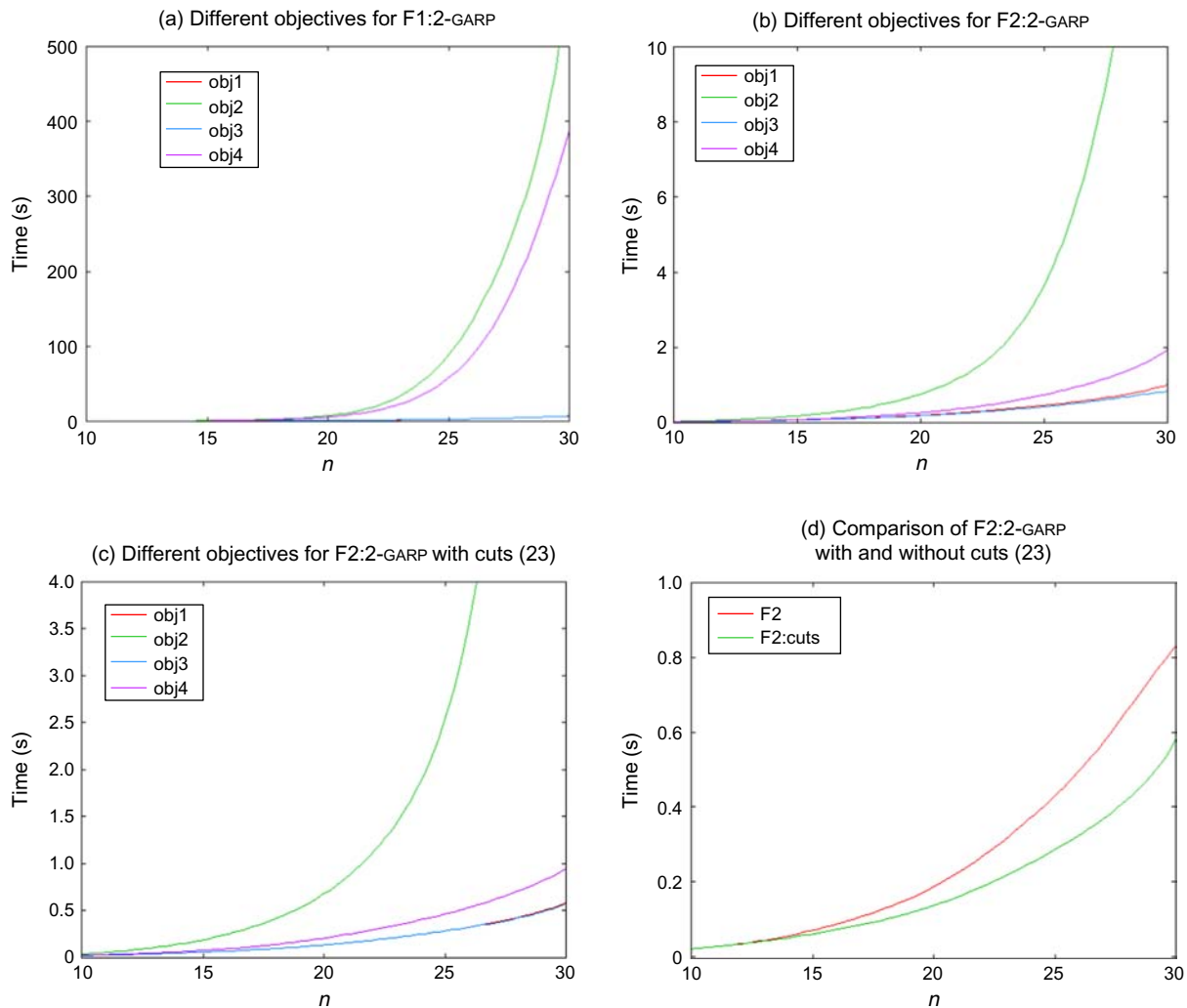
The initial solution is obtained as explained in Section 6.2 by considering 10 distinct permutations in procedure P1. We start with the identity permutation and we subsequently consider nine other randomly generated permutations. Given the preference graphs G^1 and G^2 , we use CPLEX 12.4 to check whether $\mathcal{P}(G^1, G^2)$ defines a nonempty domain.

Finally, the local search procedure $LS(\mathbf{X})$ (Algorithm 2) investigates the neighborhoods $\mathcal{N}_{st}(\mathbf{X})$ successively, starting with the arc $s \rightarrow t$ that has the smallest value $m_{st} = \max\{p'_s x_s^1 - p'_s x_t^1, p'_s x_s^2 - p'_s x_t^2\}$. It then explores the arc with the second smallest value of m_{st} , and so on. The rationale behind this choice is that the elimination of an arc $s \rightarrow t$ with the smallest value m_{st} is unlikely to create a new cycle.

7.3. Computational Results

All the algorithms presented in this paper have been coded in C using Microsoft Visual Studio C++ 2010. The experiments were run on a Dell Optiplex 760 personal computer with a Pentium R processor, 3.16 GHz clock speed and 3.21 GB RAM, equipped with Windows XP. All CPU times are expressed in *seconds*, and the plots of the average CPU times are smooth Bézier approximations (Farin 2006) of the real plots. We compare different implementations of each MIP formulation in Section 7.3.1, and we compare

Figure 1. Different implementations of the MIP formulations for 2-GARP.



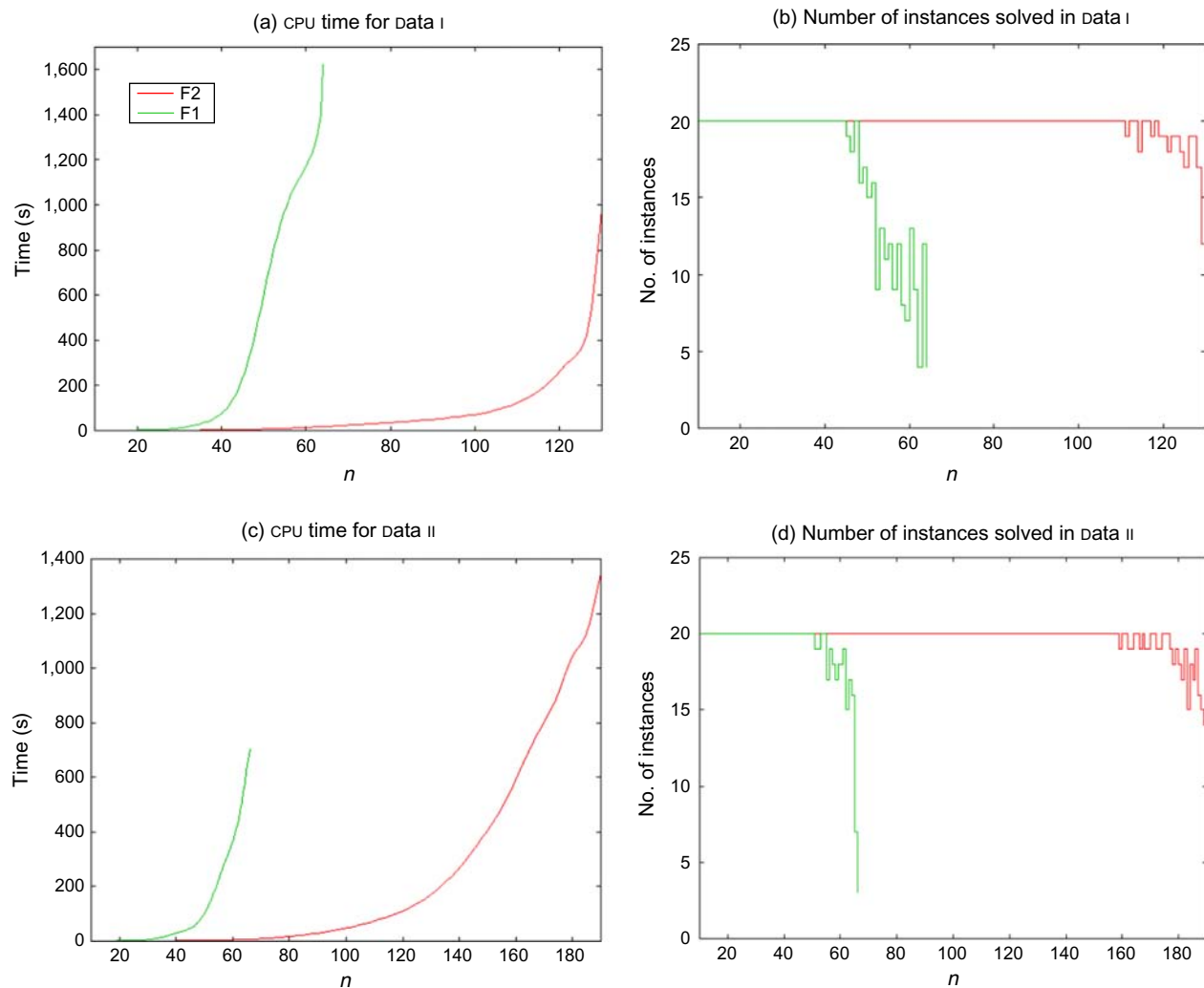
the formulations among themselves in Section 7.3.2. The best MIP formulation is compared to the SA algorithm in Section 7.3.3. Finally, we further investigate the efficiency of the SA algorithm in Section 7.3.4.

7.3.1. Comparison of Different Implementations of the MIP Formulations. We use instances from Data I with at most 30 observations to compare four different implementations of each MIP formulation. These four implementations correspond with the four different objective functions presented in the previous section.

In Figure 1 we plot the CPU time of the two MIP formulations for 2-GARP. Figure 1(a) shows the average CPU time of F1:2-GARP for each of the four objectives. The first objective (*obj1*) and the third objective (*obj3*) yield the smallest average CPU time, while the remaining two objectives (*obj2* and *obj4*) lead to considerably higher average CPU time. Figure 1(b) shows the average CPU time of F2:2-GARP for the four objectives. The objective function *obj3* leads again to smaller average CPU time than the other ones, with *obj1* a close second. Figure 1(c) displays the average

CPU time of F2:2-GARP with the cuts (23), and we observe that the objective functions *obj1* and *obj3* also dominate the other objectives. Finally, Figure 1(d) plots the best average CPU time of F2:2-GARP for *obj3*, with and without the cuts (23). Clearly, adding the cuts (23) is beneficial for the formulation F2:2-GARP. Therefore, in the rest of the section, whenever we refer to the formulation F2:2-GARP, we implicitly mean that we include the cuts (23) and that we use the objective function *obj3*. The comparison of the MIP formulations for 2-SARP yields the same results: using the objective function *obj3* and the cuts (23) leads to the best implementation among those investigated.

7.3.2. Comparison of the MIP Formulations. We now turn to a more extensive evaluation of the alternative formulations F1 and F2 for each of the problems 2-GARP and 2-SARP. In Figure 2, we compare the best implementations of F1:2-GARP and F2:2-GARP on instances from Data I and Data II. We perform a similar comparison for F1:2-SARP and F2:2-SARP in Figure 3. With respect both to the average CPU time and to the number of instances

Figure 2. (Color online) Comparison of the MIP formulations F1 and F2 for 2-GARP.

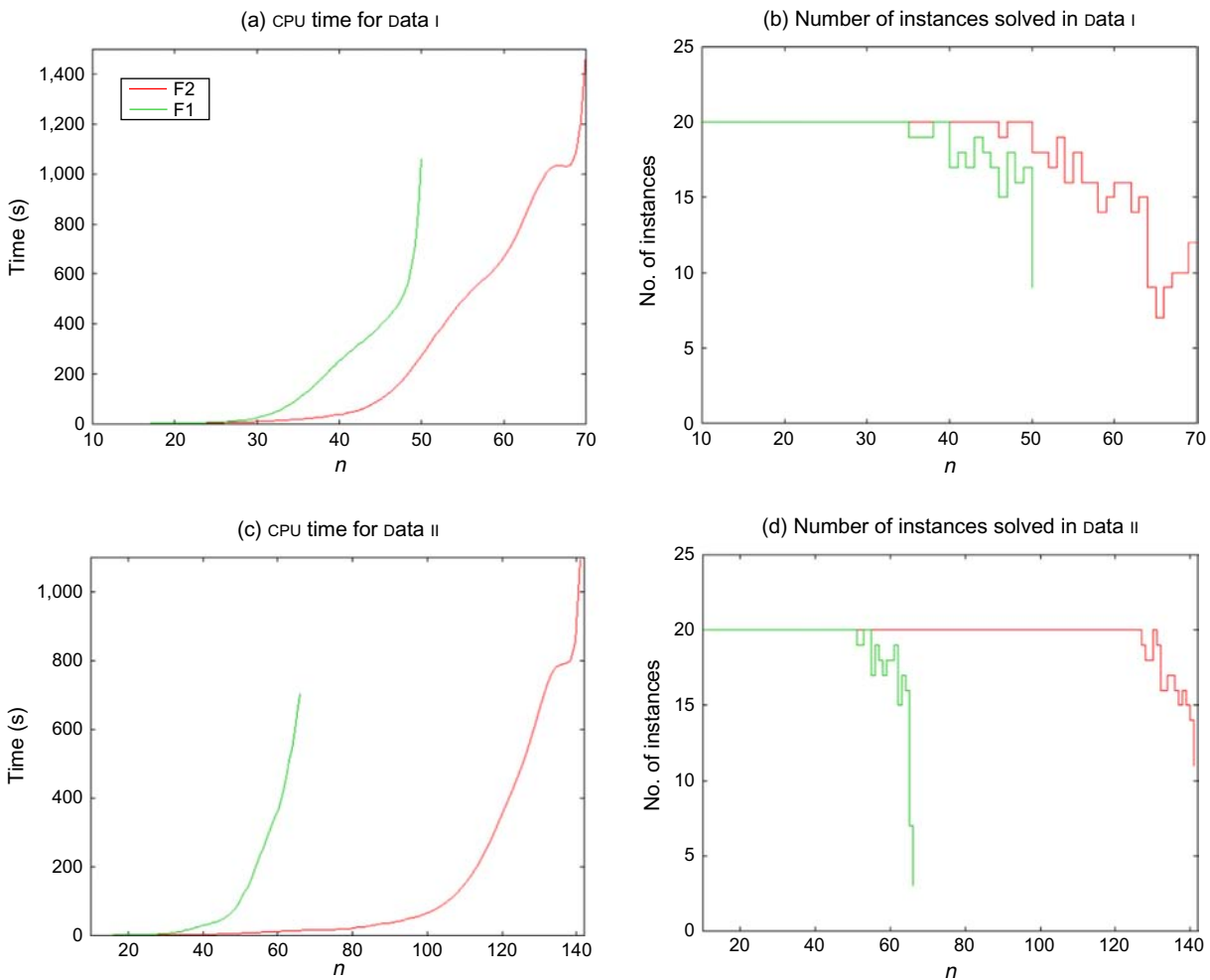
solved to optimality within the time limit (30 minutes), we find that F2:2-GARP dominates F1:2-GARP, and that F2:2-SARP dominates F1:2-SARP. On average, F2:2-GARP (respectively, F2:2-SARP) solves instances containing two to three times more observations than the instances that can be solved using F1:2-GARP (respectively, F1:2-SARP). When looking at the average CPU time, we observe that instances of 2-SARP tend to be more difficult than instances of 2-GARP with the same size. This may be due to the constraints (EC.12)–(EC.13) in formulation F1:2-SARP and to the constraints (19) in formulation F2:2-SARP, which significantly increase the size of these MIP formulations.

As a final remark, we observe from Figures 2 and 3 that the exact algorithms based on our MIP formulations can solve instances with up to 120 observations in case of 2-GARP, and 60 observations in case of 2-SARP, within 10 minutes. Conveniently, these numbers exceed the size of most data sets considered in revealed preference analyses in the literature (see, for example, Cherchye et al. 2008 and Echenique et al. 2011a). As we discuss next, our SA algorithm is effective for solving larger-sized instances.

7.3.3. Comparison of the MIP Formulations with the SA Algorithm. Figure 4 shows the average CPU time and the number of instances solved within the time limit of 30 minutes by the SA algorithm and the best MIP formulations, when applied to instances from data I and data II. For 2-GARP, the SA algorithm solves instances with size on average twice the size of instances that can be solved using F2:2-GARP, irrespective of whether the instances are drawn from data I or data II. On the other hand, for instances of 2-SARP from data II the SA algorithm solves instances with size on average twice the size of instances that can be solved using F2:2-SARP. For instances in data I, that factor increases up to four. Furthermore, for these instances the difference between the average CPU time required by the SA algorithm for 2-GARP and 2-SARP is very small; this is not the case for the MIP formulations.

Overall, the SA algorithm is effective for solving large instances of 2-GARP and 2-SARP, with or without assignable information. We mention that by ignoring the time limit when running the SA algorithm we have found that the initial real-world consumption data set (with more than

Figure 3. (Color online) Comparison of the MIP formulations F1 and F2 for 2-SARP.



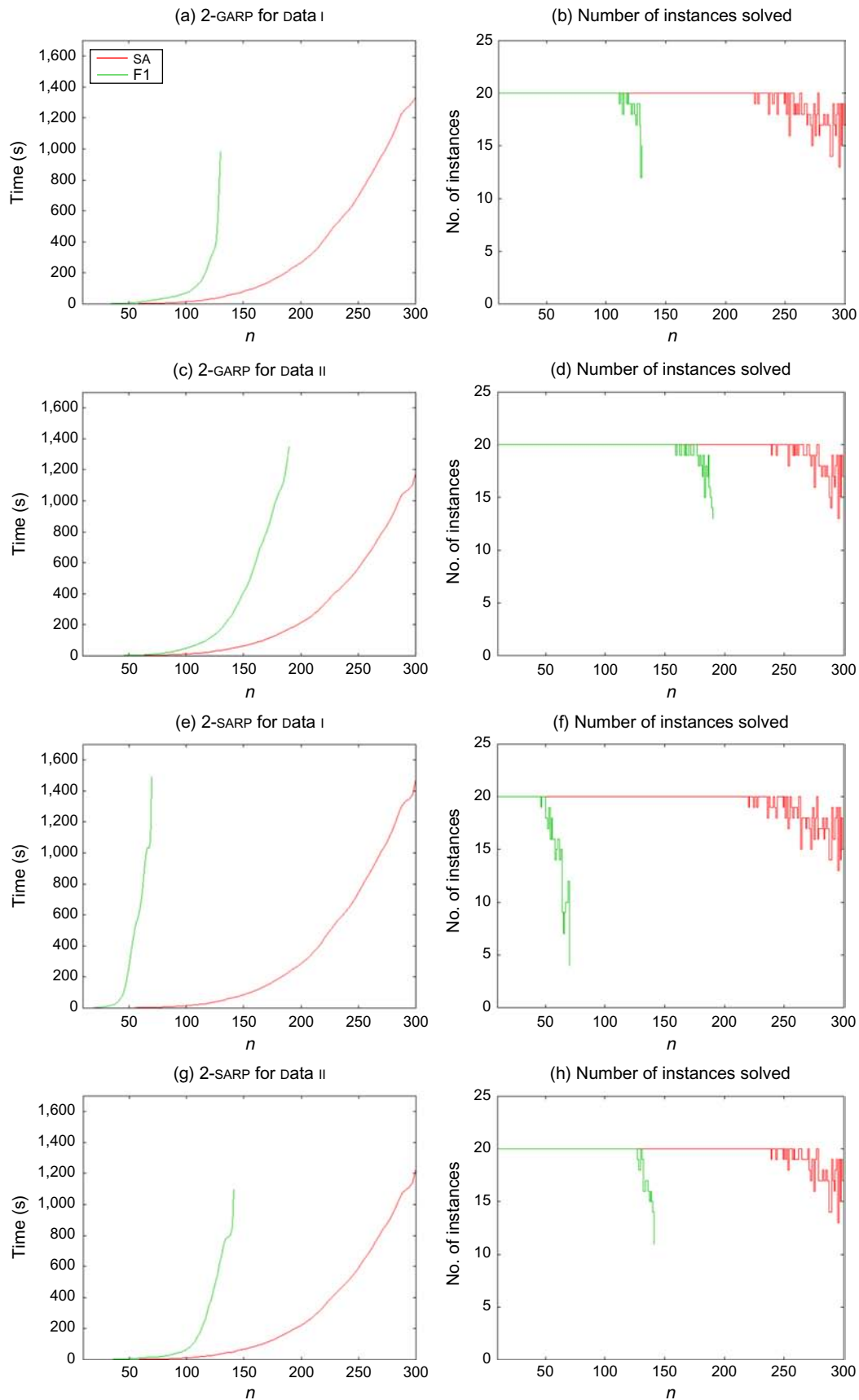
500 observations) satisfies 2-SARP, and hence 2-GARP. The good results achieved by the SA algorithm should not conceal the fact that it is a heuristic and that there may exist consumption data that are consistent with 2-GARP or 2-GARP for which the SA algorithm does not find a feasible solution (with objective function value of zero), even though such a solution exists. This phenomenon is evaluated in the next section.

7.3.4. Effectiveness of the SA Algorithm. In this section, we first evaluate the effectiveness of the SA algorithm for solving 2-GARP instances from Data III and Data IV. The assumption underlying these experiments is that the consideration of proportional assignable information will (1) increase the difficulty of the instances and (2) turn some “Yes” instances from Data I and Data II into “No” instances. We use a time limit of five minutes when applying the SA algorithm.

Figure 5 displays the average CPU time and the number of Yes instances found by the MIP formulation F2:2-GARP and the SA algorithm. From Figure 5(b), we observe that the SA algorithm identifies almost all the Yes instances in

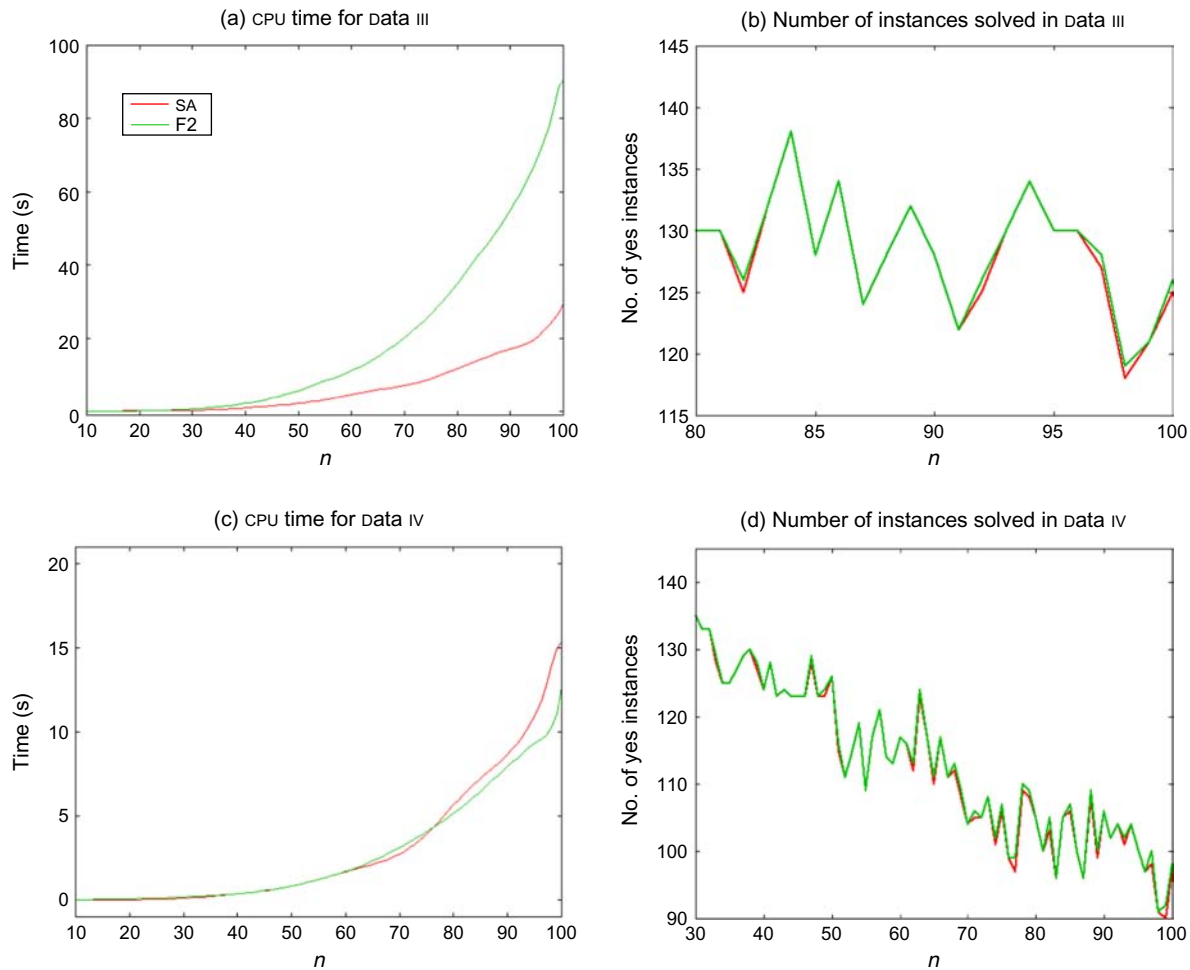
Data III, except five. The plot focuses on instances with more than 80 observations; for smaller values of n , the SA algorithm and F2:2-GARP find exactly the same number of Yes instances. The average CPU time of the SA algorithm is clearly smaller than that of F2:2-GARP as shown in Figure 5(a). For instances from Data IV, the SA algorithm fails to identify 28 Yes instances out of a total of 10,724 Yes instances. (Note that for each value of n , there are 140 instances. For two instance sizes, $n = 77$ and $n = 98$, the SA algorithm fails to identify two Yes instances. For the remaining value of n , the SA algorithm either finds the same number of Yes instances as the MIP formulation F2:2-GARP or only finds one less than the MIP formulation.) Figure 5(c) indicates that the average CPU time of the SA algorithm increases significantly, and even exceeds that of the MIP formulation F2:2-GARP for large values of n . This is mainly due to the increasing number of No instances present in Data IV, some of which force the SA algorithm to run until the time limit. Overall, we conclude from Figure 5 that the SA algorithm is efficient for identifying Yes instances of 2-GARP.

Figure 4. Comparison of the best MIP formulations and the SA algorithm.



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Figure 5. Comparison of the number of yes instances of 2-GARP found by F2 and by SA.



Similar results are reported for 2-SARP in Figure 6. In this case, F2:2-SARP could not solve most of the larger instances (say, with $n > 50$) in Data III within the time limit of five minutes. As a result, many yes instances were not correctly identified by this approach, which explains the observations in Figures 6(a)–6(b).

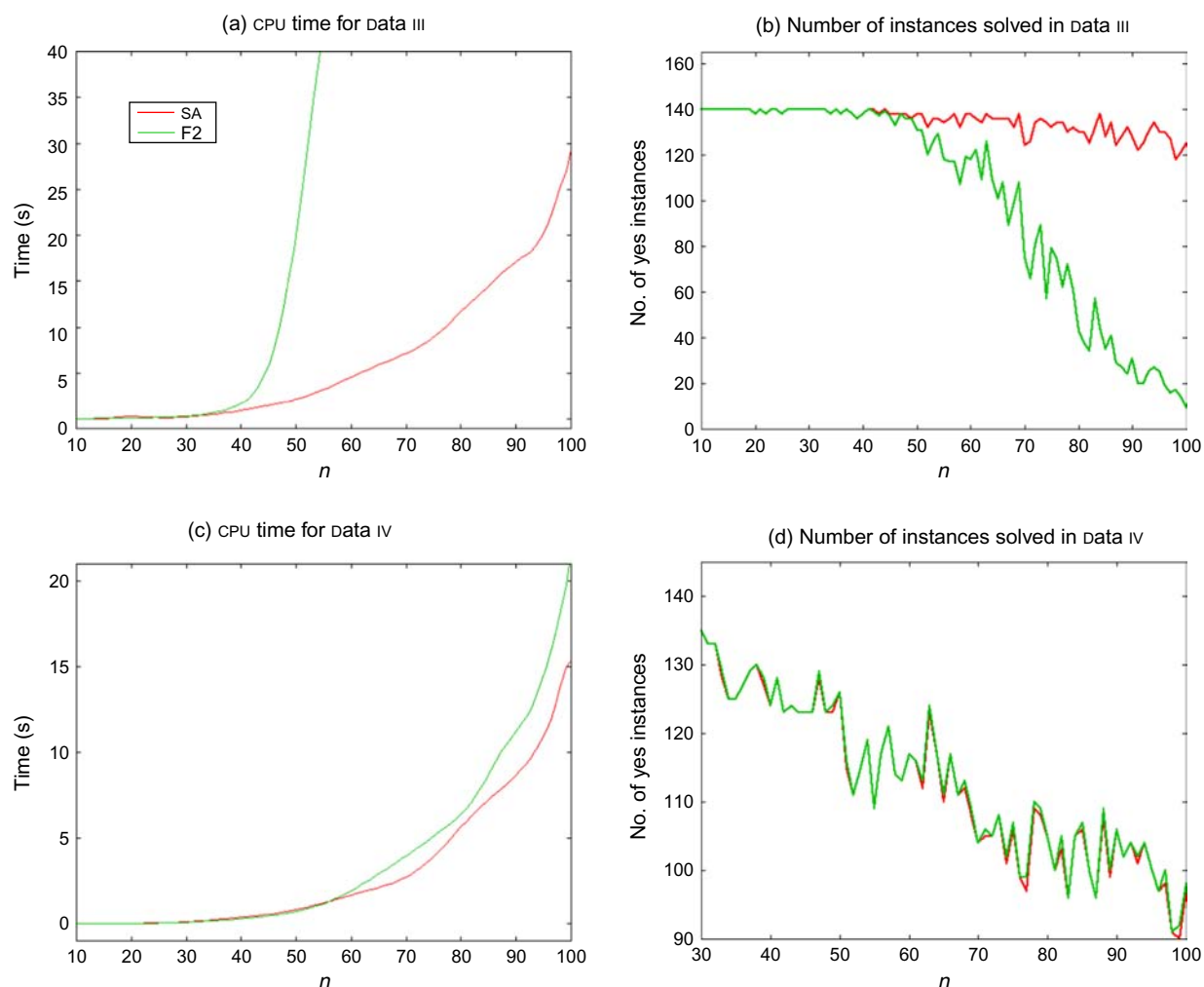
To summarize, the SA algorithm can solve much larger instances than the MIP formulations, and seldom outputs “Undecided” when “Yes” is the correct answer.

8. Conclusion

We consider the problem of testing whether household behavior involving multiple decision makers is consistent with the Generalized Axiom of Revealed Preference (GARP), and the Strong Axiom of Revealed Preference (SARP). More concretely, we consider the collective consumption models called 2-GARP and 2-SARP, which are extensions of GARP and SARP to households with two decision makers. The models allow for so-called *assignable information*, meaning that information concerning consumption of individual members of the household can be taken into account.

We prove that it is NP-complete to test these two models, even if there is no assignable information. Next, based on a particular characterization of GARP relying upon utility levels, we propose new MIP formulations for these models. The resulting formulations have significantly less constraints than the previous models from literature. Also, we propose an intricate simulated annealing (SA) heuristic, which is based on a representation of a continuous solution space as a solution space with a combinatorial structure.

Finally, to test the formulations and the SA heuristic, we have conducted extensive computational experiments. These experiments reveal the following basic insights. Firstly, the exact algorithms based on our new MIP formulations are effective for solving medium-sized instances of both 2-GARP and 2-SARP. More specifically, all instances with up to 120 observations in case of 2-GARP, and 60 observations in case of 2-SARP, are solved within 10 minutes. These results are a huge improvement compared to the results that correspond to previous models from literature. Secondly, the SA heuristic is able to solve even larger instances of our problems: instances of up to 300 observations are solved within half an hour. Moreover,

Figure 6. Comparison of the number of yes instances of 2-SARP found by F2 and by SA.

it happens only very rarely that the SA-heuristic does not identify a yes-instance as such.

Arguably, testing rationality is only a first step in the analysis of household consumption data. Next steps involve recovery and counterfactual analysis, which also raises challenging issues from a computational point of view. We anticipate that the methods and models proposed here can also play a supporting role in this other context.

Supplemental Material

Supplemental material to this paper is available at <https://doi.org/10.1287/opre.2016.1527>.

Acknowledgments

The authors would like to thank the reviewers for their constructive comments that led to an improved version of the text. Further, the project leading to these results has been partially funded by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy Office, [Grant P7/36]. Laurens Cherchye gratefully acknowledges the European Research Council (ERC) for his Consolidator Grant 614221 and the Research Fund KU

Leuven for the Grant STRT1/08/004. Bram De Rock gratefully acknowledges the European Research Council (ERC) for his Starting Grant 263707. Frits Spijksma gratefully acknowledges the FWO for Grant no. G.0447.10N.

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