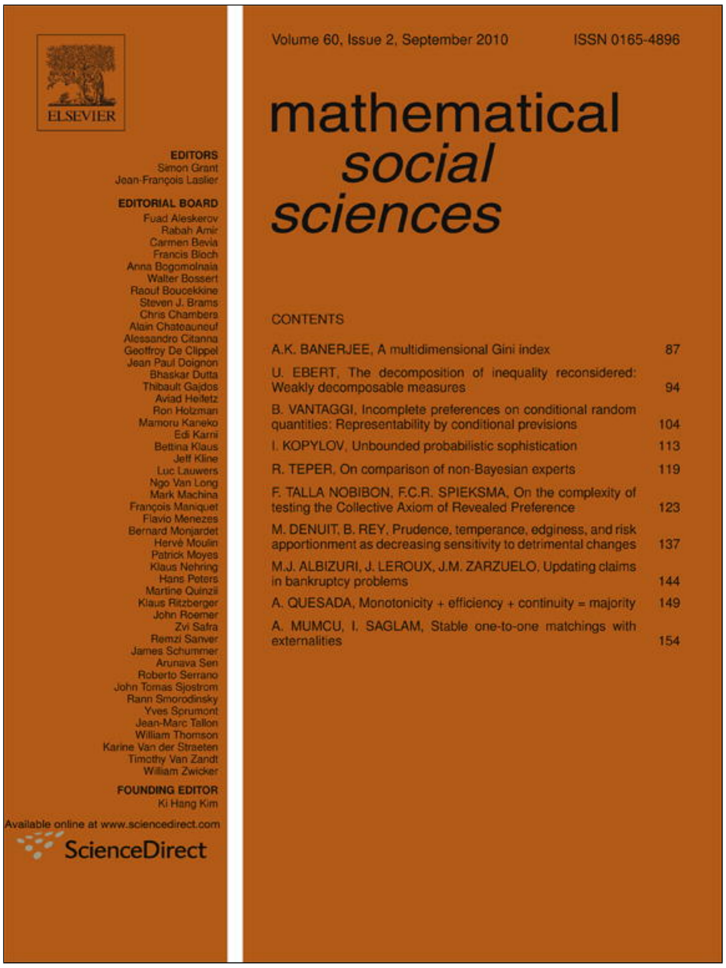
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On the complexity of testing the Collective Axiom of Revealed Preference

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ABSTRACT

We prove that the problem of testing whether data of consumption expenditures satisfy the Collective Axiom of Revealed Preference (CARP) is an NP-complete problem.

1. Introduction

In this note we investigate the problem of testing whether an observed dataset satisfies a particular micro-economic property known as the Collective Axiom of Revealed Preference. More concrete, consider a two-member household that operates in an economy with N goods. At times $t = 1, 2, \ldots, T$, the household purchases a certain quantity of each of the goods $q_t \in \mathbb{R}^N_+$ (also known as a *bundle*), at corresponding prices $p_t \in \mathbb{R}^N_{++}$. We refer to a pair of N-vectors (p_t, q_t) as an *observation*, and we call the set of observations $S = \{(p_t, q_t) : t \in \mathbb{T} \equiv \{1 \ldots, T\}\}$ the *dataset*.

It is well known that in the case of a single decision-maker, testing properties like the Weak Axiom of Revealed Preference (WARP), the Strong Axiom of Revealed Preference (SARP) and the Generalized Axiom of Revealed Preference (GARP) can be done efficiently (Varian, 1982). Here, we show that testing the Collective Axiom of Revealed Preference for a dataset originating from a household consisting of two decision-makers, leads to a computationally difficult problem.

The Collective Axiom of Revealed Preference (CARP) provides a testable, nonparametric, necessary and sufficient condition for a collective rationalization of the dataset. CARP was introduced in Cherchye et al. (2007); we refer to Cherchye et al. (2008) and Talla Nobibon et al. (forthcoming) and the references therein for detailed discussions of CARP. In Section 2 we state the rules defining CARP. The purpose of this note is to show that testing whether a given dataset S satisfies CARP is NP-complete and this is done in Section 3; we conclude in Section 4.

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2. Problem description and notation

Following micro-economic theory, we assume that each of the two members of the household has a (hypothetical) preference relation over bundles. This preference relation is denoted by H_0^1 for member 1, and H_0^2 for member 2. Furthermore, the phrase "$(q_s, q_t) \in H_0^i$" means that we hypothesize that member i (directly) prefers the bundle q_s over the bundle q_t; for $i \in \{1, 2\}$ and $s, t \in \mathbb{T}$. Notice that (q_s, q_t) is an ordered pair. Next, H^i $(i \in \{1, 2\})$ represents the transitive closure of H_0^i, that is $(q_s, q_t) \in H^i$ means that there exists a (possibly empty) sequence $u, v, \ldots, z \in \mathbb{T}$ with $(q_s, q_u) \in H_0^i$, $(q_u, q_v) \in H_0^i$,..., and $(q_z, q_t) \in H_0^i$. Thus given H_0^i for $i \in \{1, 2\}$, the transitive closures H^i follow. For ease of exposition, the scalar product $p_t \cdot q_t$ (which is the amount of money spent in observation t) is written as $p_t q_t$.

Given this notion of hypothetical preference relations, CARP is defined as follows (see Cherchye et al. (2007)).

Definition 1 (*CARP*). Given is a dataset $S = \{(p_t, q_t) : t \in \mathbb{T}\}$. S satisfies CARP if there exist hypothetical relations H_0^1 and H_0^2 that satisfy for all $s, t, t_1, t_2 \in \mathbb{T}$:

Rule 1: if $p_s q_s \geq p_s q_t$ then either $(q_s, q_t) \in H_0^1$ or $(q_s, q_t) \in H_0^2$;
Rule 2: if $p_s q_s \geq p_s q_t$ and $(q_t, q_s) \in H^m$ then $(q_s, q_t) \in H_0^\ell$ with $\ell \neq m$;
Rule 3: if $p_s q_s \geq p_s(q_{t_1} + q_{t_2})$ and $(q_{t_1}, q_s) \in H^m$ then $(q_s, q_{t_2}) \in H_0^\ell$ with $\ell \neq m$;
Rule 4: if $p_s q_s > p_s q_t$ then either $(q_t, q_s) \notin H^1$ or $(q_t, q_s) \notin H^2$;
Rule 5: if $p_s q_s > p_s(q_{t_1} + q_{t_2})$ then either $(q_{t_1}, q_s) \notin H^1$ or $(q_{t_2}, q_s) \notin H^2$.

An inequality of the form $p_s q_s \geq p_s(q_{t_1} + q_{t_2})$ (or of the form $p_s q_s > p_s(q_{t_1} + q_{t_2})$) is called a *double-sum inequality*.

Rule 1 states that, if the bundle q_s was chosen while the bundle q_t was equally attainable (under the prices p_s), then it must be that at least one member prefers the bundle q_s over the bundle q_t (i.e. $(q_s, q_t) \in H_0^1$ or $(q_s, q_t) \in H_0^2$). Rule 2 states that, if member m prefers q_t over q_s while the bundle q_t is not more expensive than q_s against prices p_s (i.e. $p_s q_s \geq p_s q_t$), then member ℓ prefers q_s over q_t. Rule 3 states that, if the summed bundle $q_{t_1} + q_{t_2}$ is attainable and member m prefers q_{t_1} over q_s, then the other member (member ℓ) prefers q_s over q_{t_2}. Rule 4 states that, if q_t was cheaper when q_s was chosen, then it cannot be that both members prefer q_t over q_s. Finally, Rule 5 states that, if $q_{t_1} + q_{t_2}$ was cheaper when q_s was chosen, then it cannot be that one member prefers q_{t_1} over q_s while, at the same time, the other member prefers q_{t_2} over q_s.

The problem of testing whether a given dataset S satisfies CARP can be phrased as the following decision problem.
INSTANCE: A dataset $S = \{(p_t, q_t) : t \in \mathbb{T}\}$.
QUESTION: Does the dataset satisfies CARP? In other words, do there exist H_0^1 and H_0^2 such that *Rules* 1–5 are satisfied?
In the next section, we prove that the problem of testing CARP is NP-complete.

3. Complexity result

In this section we prove that testing CARP is NP-complete. The proof uses a reduction from the Not-All-Equal-3Sat problem, which is defined as follows.
INSTANCE: Set $X = \{x_1, \ldots, x_n\}$ of n variables, collection $C = \{C_1, \ldots, C_m\}$ of m clauses over X such that each clause $C_\ell \in C$ has $|C_\ell| = 3$.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?
Garey and Johnson (1979) proved that the Not-All-Equal-3Sat problem is NP-complete.

In the proof, we consider instances of the Not-All-Equal-3Sat problem where no variable occurs more than once in the same clause. This is without loss of generality, since, given an instance of Not-All-Equal-3Sat where a clause contains the same variable twice, we can simplify that clause to get a clause with two distinct variables. By appropriately adding a new variable and a new clause, we can transform that instance into an instance of the Not-All-Equal-3Sat problem where no variable occurs more than once in the same clause. As illustration, the clause $(x_1 \vee x_2 \vee x_2)$ can be replaced by $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3)$.

The idea behind the proof is the following: for each variable and for each clause of the Not-All-Equal-3Sat instance, we build a set of observations. Each of these observations concerns a number of goods; in particular, we have a price-vector, and a quantity-vector for each observation. By choosing appropriate values for the prices and the quantities, we establish for each pair and for each triple of observations the inequality desired. Next, the implications for the hypothetical relations H_0^1 and H_0^2 induced by *Rules* 1–5 are such that their existence is equivalent to the instance of the Not-All-Equal-3Sat problem being satisfiable.

Our result:

Theorem 1. *Testing whether a given dataset S satisfies CARP is NP-complete.*

The proof of this theorem is structured as follows. First, we build a dataset S given the instance of Not-All-Equal-3Sat. Next, we enumerate for each pair of observations (s, t) and for each triple of observations (s, t_1, t_2) whether an inequality of the form $p_s q_s \geq p_s q_t$ (or $p_s q_s > p_s q_t$), or of the form $p_s q_s \geq p_s(q_{t_1} + q_{t_2})$ (or $p_s q_s > p_s(q_{t_1} + q_{t_2})$) is present. This is described in Claims 1 and 2. Third, we argue the equivalence of a yes-instance of Not-All-Equal-3Sat and the dataset S satisfying CARP. For the sake of simplicity, throughout this text we will also call $t \in \mathbb{T}$ an observation while referring to (p_t, q_t).

Notice that it is not hard to see that the problem of testing CARP is in the class NP: given the relations H_0^1 and H_0^2; (and hence H^1 and H^2) we simply check, for each pair or triple of observations, whether *Rules* 1–5 hold. Clearly, this can be done in polynomial time.

In the first step of the proof, we aim at building the dataset S. We shall first determine the set $\mathbb{T}$ of indices of observations. Next, we derive the number of goods in the economy and finally, for each observation, we derive a vector containing the price (respectively quantity) of each good for that observation.

Consider an arbitrary instance of the Not-All-Equal-3Sat problem where no variable occurs more than once in the same clause. We build the set of observations as follows. For each variable $x_i \in X$ ($i = 1, \ldots, n$), we have two observations specified by x_i and $\bar{x}_i$, where the latter refers to the negation of x_i. We define $\mathbb{T}_1 = \{x_i, \bar{x}_i : i = 1, \ldots, n\}$ with cardinality $|\mathbb{T}_1| = 2n$. The observations in $\mathbb{T}_1$ are called *variable observations*.

For each clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell) \in C$, where the literal χ_1^ℓ is either the variable x_i or its negation $\bar{x}_i$, χ_2^ℓ is either x_j or $\bar{x}_j$, and χ_3^ℓ is either x_k or $\bar{x}_k$ with $1 \leq i < j < k \leq n$ (this ordering of indices can be achieved by permuting some literals), we define six observations $\mathbb{T}_2^\ell = \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$. The first three observations in $\mathbb{T}_2^\ell$ are called *literal observations*. The last three observations in $\mathbb{T}_2^\ell$ are *associated observations*; each associated observation is associated to a literal observation. In particular, t_1^ℓ is associated with χ_1^ℓ, t_2^ℓ with χ_2^ℓ and t_3^ℓ with χ_3^ℓ. Let $\mathbb{T}_2 = \cup_{l=1}^m \mathbb{T}_2^\ell$ with $|\mathbb{T}_2| = 6m$. The observations in $\mathbb{T}_2$ are called *clause observations*. That is, a clause observation is either a literal observation or an associated observation. In total, the set of observations $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$ contains $T = |\mathbb{T}| = 2n + 6m$ observations.

To illustrate the reduction, we consider the following example of Not-All-Equal-3Sat problem, subsequently referred to as *the example*. The set of variables is $X = \{x_1, x_2, x_3\}$, and there are two clauses $C_1 = (x_1 \vee x_2 \vee x_3)$ and $C_2 = (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$; that is $\chi_1^1 = x_1$, $\chi_2^1 = x_2$, $\chi_3^1 = x_3$, $\chi_1^2 = \bar{x}_1$, $\chi_2^2 = x_2$ and $\chi_3^2 = \bar{x}_3$. Notice that the truth assignment $x_1 = x_2 = 1$ and $x_3 = 0$ is a solution to this Not-All-Equal-3Sat instance. For the example, the variable observations are $\{x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3\}$ while the clause observations are $\{\chi_1^1, \chi_2^1, \chi_3^1, t_1^1, t_2^1, t_3^1\}$ for the first clause, and $\{\chi_1^2, \chi_2^2, \chi_3^2, t_1^2, t_2^2, t_3^2\}$ for the second clause. The reduction leads to a set of observations $\mathbb{T} = \{x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3, \chi_1^1, \chi_2^1, \chi_3^1, t_1^1, t_2^1, t_3^1, \chi_1^2, \chi_2^2, \chi_3^2, t_1^2, t_2^2, t_3^2\}$ with 18 elements.

To further describe the dataset S, we need to fix the number of goods in each bundle, and for each observation in $\mathbb{T}$, we must specify the price and the quantity of each good. We consider an economy with $N = 2T^2$ goods. We now specify the price and the quantity of the N goods for each observation in $\mathbb{T}$. For ease of exposition, a bundle of $N = 2T^2$ goods is represented by two blocks, each block being a $T \times T$ matrix. Each cell in each block represents a good.

We index the rows and columns of the first $T \times T$ matrix (referred to as Block 1 in the rest of this text) by the observations in $\mathbb{T}$. We use, both for the rows and for the columns, the following ordering: $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n, \chi_1^1, \chi_2^1, \chi_3^1, t_1^1, t_2^1, t_3^1, \chi_1^2, \chi_2^2, \chi_3^2, t_1^2, t_2^2, t_3^2, \ldots, \chi_1^m, \chi_2^m, \chi_3^m, t_1^m, t_2^m, t_3^m$. For the example, Block 1 is represented in Table 1.

We use the same indices for naming the rows and columns of the second $T \times T$ matrix (subsequently called Block 2 throughout this text). Hence, we can identify a good by specifying a pair (s, t) where s is the row-index (an observation), and where t is the

Table 1
Block 1 for the example.

	x_1	$\bar{x}_1$	x_2	$\bar{x}_2$	x_3	$\bar{x}_3$	χ_1^1	χ_2^1	χ_3^1	t_1^1	t_2^1	t_3^1	χ_1^2	χ_2^2	χ_3^2	t_1^2	t_2^2	t_3^2
x_1																		
$\bar{x}_1$																		
x_2																		
$\bar{x}_2$																		
x_3																		
$\bar{x}_3$																		
χ_1^1																		
χ_2^1																		
χ_3^1																		
t_1^1																		
t_2^1																		
t_3^1																		
χ_1^2																		
χ_2^2																		
χ_3^2																		
t_1^2																		
t_2^2																		
t_3^2																		

column-index (also an observation), and by specifying the block (either Block 1 or Block 2).

For each variable $x_i \in X$, we define

$$\Gamma_{x_i} = \{\ell \in \{1, \ldots, m\} : \text{ clause } C_\ell \in C \text{ contains literal } x_i\}.$$

Similarly,

$$\Gamma_{\bar{x}_i} = \{\ell \in \{1, \ldots, m\} : \text{ clause } C_\ell \in C \text{ contains literal } \bar{x}_i\}.$$

Further, let

$$\Delta = 1 + \max\{8, \max\{2|\Gamma_{x_i}| + 4 : i = 1, \ldots, n\}, \max\{2|\Gamma_{\bar{x}_i}| + 4 : i = 1, \ldots, n\}\},$$

(where $|A|$ is the cardinality of A). For the example, we have $\Gamma_{x_1} = \{1\}$ as x_1 appears only in the clause C_1, $\Gamma_{\bar{x}_1} = \{2\}$ because $\bar{x}_1$ is present only in C_2, $\Gamma_{x_2} = \{1, 2\}$, $\Gamma_{\bar{x}_2} = \emptyset$, $\Gamma_{x_3} = \{1\}$, $\Gamma_{\bar{x}_3} = \{2\}$ and $\Delta = 9$. Since we want to avoid prices equal to 0, we use, in the rest of this text, ε to denote a very small, strictly positive, real number.

Next, for each observation in $\mathbb{T}$ we will determine the price, as well as the quantity of each good. We will do this by distinguishing eight types of observations:

- variable observations corresponding to positive (negative) literals. The vector of prices for that observation is denoted by p_{x_i} ($p_{\bar{x}_i}$), and the bundle (purchased quantities) is denoted by q_{x_i} ($q_{\bar{x}_i}$); for $i = 1, \ldots, n$.
- clause observations corresponding to the first (second, third) literal. The vector of prices for that observation is denoted by $p_{\chi_1^\ell}$ ($p_{\chi_2^\ell}$, $p_{\chi_3^\ell}$), and the bundle by $q_{\chi_1^\ell}$ ($q_{\chi_2^\ell}$, $q_{\chi_3^\ell}$); for $\ell = 1, \ldots, m$.
- associated observations corresponding to the first (second, third) literal. The vector of prices for that observation is denoted by $p_{t_1^\ell}$ ($p_{t_2^\ell}$, $p_{t_3^\ell}$), and the bundle by $q_{t_1^\ell}$ ($q_{t_2^\ell}$, $q_{t_3^\ell}$); for $\ell = 1, \ldots, m$.

Choosing the particular values of the prices and the quantities is done with the objective of satisfying some inequalities for pairs or triples of observations. In fact, for each pair of observations (s, t), there are two goods: one in Block 1 and one in Block 2. The good in Block 1 is used to ensure that the desired inequality between $p_s q_s$ and $p_s q_t$ holds. The good in Block 2 is used to enforce the presence or absence of a double-sum inequality involving $p_s q_s$ and $p_s q_t$. All this is achieved by choosing appropriate values for the price and the quantity of each good.

We now continue by describing how the prices of all goods for all observations are set. That is, for each cell in each of the two blocks forming the set of all goods, we fix a strictly positive real value, representing the price. To achieve this, we proceed as follows. For each of the two blocks, we specify the structure of the corresponding matrix by giving a value to each cell representing the price of the good corresponding to that cell. We do this for every observation in $\mathbb{T}$.

Specifying $p_{x_i}, p_{\bar{x}_i}, p_{\chi_1^\ell}, p_{\chi_2^\ell}, p_{\chi_3^\ell}, p_{t_1^\ell}, p_{t_2^\ell}, p_{t_3^\ell}$ for goods corresponding to cells in Block 1.

For each observation $s \in \mathbb{T}$, there is a row in Block 1 indexed by s. We set the price of each good corresponding to a cell in this row equal to 1, except for the price of the good corresponding to cell (s, s): its price equals 2. The goods corresponding to the remaining cells in Block 1 get the price ε.

As an illustration, consider the observation $\bar{x}_2$ of the example. The price of goods corresponding to cells in Block 1 is given by Table 2.

Specifying p_{x_i} for goods corresponding to cells in Block 2.

Given a clause C_ℓ that contains $\bar{x}_i$ (negation of x_i), let r denote the position of $\bar{x}_i$ in the clause C_ℓ. Of course, $r \in \{1, 2, 3\}$ (notice that r depends upon ℓ and i; for reasons of convenience we simply write r instead of $r(i, \ell)$). Thus, for each clause C_ℓ with $\ell \in \Gamma_{\bar{x}_i}$, there is an associated observation t_r^ℓ in $\mathbb{T}$. The price of the good corresponding to cell $(\bar{x}_i, t_r^\ell)$ equals $\frac{1}{2|\Gamma_{\bar{x}_i}|}$. Also, the price of the good corresponding to cell $(t_r^\ell, \bar{x}_i)$ equals $\frac{1}{2|\Gamma_{\bar{x}_i}|}$. This is done for each clause C_ℓ with $\ell \in \Gamma_{\bar{x}_i}$. Notice that in total, we have $2|\Gamma_{\bar{x}_i}|$ cells with value $\frac{1}{2|\Gamma_{\bar{x}_i}|}$ in this block (Block 2). The goods corresponding to the remaining cells in Block 2 get the price ε.

Table 2
Price of goods corresponding to cells in Block 1 for observation $\bar{x}_2$.

	x_1	$\bar{x}_1$	x_2	$\bar{x}_2$	x_3	$\bar{x}_3$	χ_1^1	χ_2^1	χ_3^1	t_1^1	t_2^1	t_3^1	χ_1^2	χ_2^2	χ_3^2	t_1^2	t_2^2	t_3^2
x_1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
$\bar{x}_1$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
x_2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
$\bar{x}_2$	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x_3	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
$\bar{x}_3$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε.
χ_1^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_2^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_3^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_1^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_2^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_3^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_1^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_2^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_3^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_1^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_2^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_3^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε

As an illustration, consider the observation x_2 of the example. Since $\Gamma_{\bar{x}_2} = \emptyset$, the price of all goods corresponding to cells in Block 2 is ε.

Specifying $p_{\bar{x}_i}$ for goods corresponding to cells in Block 2.

We use an approach similar to the one used to determine p_{x_i}. Now, let r denote the position of x_i in the clause C_ℓ. For each clause C_ℓ with $\ell \in \Gamma_{x_i}$, there is an associated observation t_r^ℓ. The price of the good corresponding to cell (x_i, t_r^ℓ) equals $\frac{1}{2|\Gamma_{x_i}|}$. Also, the price of the good corresponding to cell (t_r^ℓ, x_i) equals $\frac{1}{2|\Gamma_{x_i}|}$. This is done for each clause C_ℓ with $\ell \in \Gamma_{x_i}$. The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the observation $\bar{x}_2$ of the example. The prices of goods corresponding to cells in Block 2 are given in Table 3. Notice that there are four goods with price $\frac{1}{4}$.

Specifying $p_{\chi_1^\ell}$ for goods corresponding to cells in Block 2.

There are two goods corresponding to cells in Block 2 that have a price different from ε. These are the goods corresponding to the two cells (χ_3^ℓ, t_3^ℓ) and (t_3^ℓ, χ_3^ℓ); the price for these goods equals $\frac{1}{2}$. The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ we have $\chi_1^\ell = x_1$ and the two goods with price $\frac{1}{2}$ correspond to cells (χ_3^1, t_3^1) and (t_3^1, χ_3^1). The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells (χ_3^2, t_3^2) and (t_3^2, χ_3^2) get the price $\frac{1}{2}$; the goods corresponding to the remaining cells in Block 2 get the price ε.

Specifying $p_{\chi_2^\ell}$ for goods corresponding to cells in Block 2.

Again, there are two goods that have price $\frac{1}{2}$, namely those corresponding to the cells (χ_1^ℓ, t_1^ℓ) and (t_1^ℓ, χ_1^ℓ). The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ we have $\chi_2^\ell = x_2$ and the two goods with price $\frac{1}{2}$ correspond to cells (χ_1^1, t_1^1) and (t_1^1, χ_1^1). The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells (χ_1^2, t_1^2) and (t_1^2, χ_1^2) get the price $\frac{1}{2}$; the goods corresponding to the remaining cells in Block 2 get the price ε.

Specifying $p_{\chi_3^\ell}$ for goods corresponding to cells in Block 2.

Also here, there are two goods with price $\frac{1}{2}$, namely those corresponding to the cells (χ_2^ℓ, t_2^ℓ) and (t_2^ℓ, χ_2^ℓ). The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ we have $\chi_3^\ell = x_3$ and the two goods with price $\frac{1}{2}$ correspond to cells (χ_2^1, t_2^1) and (t_2^1, χ_2^1). The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells (χ_2^2, t_2^2) and (t_2^2, χ_2^2) get the price $\frac{1}{2}$; the goods corresponding to the remaining cells in Block 2 get the price ε.

Specifying $p_{t_1^\ell}$ for goods corresponding to cells in Block 2.

Recall that the observation t_1^ℓ is associated with the literal observation χ_1^ℓ. Further, the literal χ_1^ℓ is either x_i or $\bar{x}_i$ for a given $i \in \{1, \ldots, n\}$. In both cases, there are only two goods corresponding to cells in Block 2 that have price different from ε.

If $\chi_1^\ell = x_i$, then the two goods of Block 2 with price $\frac{1}{2}$ are those corresponding to cells $(\chi_2^\ell, \bar{x}_i)$ and $(\bar{x}_i, \chi_2^\ell)$. The goods corresponding to the remaining cells in Block 2 get the price ε. On the other hand, if $\chi_1^\ell = \bar{x}_i$, then the two goods of Block 2 corresponding to cells (χ_2^ℓ, x_i) and (x_i, χ_2^ℓ) have price $\frac{1}{2}$. The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ the observation t_1^1 is such that the goods corresponding to cells $(\chi_2^1, \bar{x}_1)$ and $(\bar{x}_1, \chi_2^1)$ in Block 2 have price $\frac{1}{2}$ since $\chi_1^1 = x_1$. The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells (χ_2^2, x_1) and (x_1, χ_2^2) in Block 2 have price $\frac{1}{2}$ for observation t_1^2 because $\chi_1^2 = \bar{x}_1$. The goods corresponding to the remaining cells in Block 2 get the price ε.

Specifying $p_{t_2^\ell}$ for goods corresponding to cells in Block 2.

The observation t_2^ℓ is associated with χ_2^ℓ which is either x_j or $\bar{x}_j$ for a given $j \in \{1, \ldots, n\}$.

Table 3
Price of goods corresponding to cells in Block 2 for observation $\bar{x}_2$.

	x_1	$\bar{x}_1$	x_2	$\bar{x}_2$	x_3	$\bar{x}_3$	χ_1^1	χ_2^1	χ_3^1	t_1^1	t_2^1	t_3^1	χ_1^2	χ_2^2	χ_3^2	t_1^2	t_2^2	t_3^2
x_1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
$\bar{x}_1$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
x_2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	$\frac{1}{4}$	ε	ε	ε	ε	ε	$\frac{1}{4}$	ε
$\bar{x}_2$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
x_3	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
$\bar{x}_3$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_1^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_2^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_3^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_1^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_2^1	ε	ε	$\frac{1}{4}$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_3^1	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_1^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_2^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
χ_3^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_1^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_2^2	ε	ε	$\frac{1}{4}$	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε
t_3^2	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε	ε

If $\chi_2^\ell = x_j$, then the two goods of Block 2 with price $\frac{1}{2}$ are those corresponding to cells $(\chi_3^\ell, \bar{x}_j)$ and $(\bar{x}_j, \chi_3^\ell)$. The goods corresponding to the remaining cells in Block 2 get the price ε.
Otherwise, if $\chi_2^\ell = \bar{x}_j$ then the two goods of Block 2 corresponding to cells (χ_3^ℓ, x_j) and (x_j, χ_3^ℓ) have price $\frac{1}{2}$. The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ the observation t_2^1 is such that the goods corresponding to cells $(\chi_3^1, \bar{x}_2)$ and $(\bar{x}_2, \chi_3^1)$ in Block 2 have price $\frac{1}{2}$ since $\chi_2^1 = x_2$. The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells $(\chi_3^2, \bar{x}_2)$ and $(\bar{x}_2, \chi_3^2)$ in Block 2 have price $\frac{1}{2}$ for observation t_2^2 because $\chi_2^2 = x_2$. The goods corresponding to the remaining cells in Block 2 get the price ε.

Specifying $p_{t_3^\ell}$ for goods corresponding to cells in Block 2.

The observation t_3^ℓ is associated with χ_3^ℓ which is either x_k or $\bar{x}_k$ for a given $k \in \{1, \ldots, n\}$.

If $\chi_3^\ell = x_k$, the two goods of Block 2 with price $\frac{1}{2}$ are those corresponding to cells $(\chi_1^\ell, \bar{x}_k)$ and $(\bar{x}_k, \chi_1^\ell)$. The goods corresponding to the remaining cells in Block 2 get the price ε.
If, on the other hand, $\chi_3^\ell = \bar{x}_k$ then the two goods of Block 2 corresponding to cells (χ_1^ℓ, x_k) and (x_k, χ_1^ℓ) have price $\frac{1}{2}$. The goods corresponding to the remaining cells in Block 2 get the price ε.

As an illustration, consider the example. For $\ell = 1$ the observation t_3^1 is such that the goods corresponding to cells $(\chi_1^1, \bar{x}_3)$ and $(\bar{x}_3, \chi_1^1)$ in Block 2 have price $\frac{1}{2}$ since $\chi_3^1 = x_3$. The goods corresponding to the remaining cells in Block 2 get the price ε. For $\ell = 2$, the goods corresponding to cells (χ_1^2, x_3) and (x_3, χ_1^2) in Block 2 have price $\frac{1}{2}$ for observation t_3^2 because $\chi_3^2 = \bar{x}_3$. The goods corresponding to the remaining cells in Block 2 get the price ε.

This achieves the description of prices: for each observation in $\mathbb{T}$ we have specified the price of every good in the bundle (that is every cell in the two blocks). It remains to fix for each observation in $\mathbb{T}$ the quantity used for each good in the bundle.

Specifying q_{x_i} for goods corresponding to cells in Block 1.

There is a row and a column in Block 1 indexed by x_i. All the goods corresponding to cells of Block 1 other than those corresponding to cells in row x_i and in column x_i get the value 0 as their quantity. As for cells in row x_i and column x_i, the good corresponding to cell (x_i, x_i) gets the value 1. The quantity of the good corresponding to cell $(x_i, \bar{x}_i)$ equals 1, and the quantity of the good corresponding to cell $(\bar{x}_i, x_i)$ equals the value $|\Gamma_{x_i}| + 1$. Moreover, for every clause C_ℓ containing $\bar{x}_i$ ($\ell \in \Gamma_{\bar{x}_i}$), the good corresponding to cell (x_i, t_r^ℓ) gets the value 1, where r denotes the position of $\bar{x}_i$ in the clause C_ℓ. The quantity of the good corresponding to cell (t_r^ℓ, x_i) equals 2. The remaining goods corresponding to cells in row x_i are not used and get the quantity 0, while those remaining in column x_i get the value Δ. Observe that a good corresponding to a cell (x_i, t) in row x_i has a non-zero value if and only if the corresponding to the cell (t, x_i) in column x_i has a value different from (more precisely less than) Δ.

As an illustration, consider the observation x_1 of the example. The quantity of goods corresponding to cells in Block 1 are given in Table 4.

Specifying $q_{\bar{x}_i}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 other than those in row $\bar{x}_i$ and in column $\bar{x}_i$ get the value 0 as their quantity. The good corresponding to cell $(\bar{x}_i, \bar{x}_i)$ has quantity 1. Also, the good corresponding to cell $(\bar{x}_i, x_i)$ has quantity 1, and the good corresponding to cell $(x_i, \bar{x}_i)$ has quantity $|\Gamma_{\bar{x}_i}| + 1$. For every clause C_ℓ containing x_i ($\ell \in \Gamma_{x_i}$), the good corresponding to cell $(\bar{x}_i, t_r^\ell)$ has quantity 1 (where r refers to the position of x_i in clause C_ℓ). The good corresponding to cell $(t_r^\ell, \bar{x}_i)$ has quantity 2. For the goods corresponding to the remaining cells in row $\bar{x}_i$, their quantity is 0, while the goods corresponding to the remaining cells in column $\bar{x}_i$ have quantity Δ.

As an illustration, consider the observation $\bar{x}_2$ of the example. The quantity of goods corresponding to cells in Block 1 are given in Table 5.

Table 4
Quantity of goods corresponding to cells in Block 1 for observation x_1.

	x_1	$\bar{x}_1$	x_2	$\bar{x}_2$	x_3	$\bar{x}_3$	x_1^1	x_2^1	x_3^1	t_1^1	t_2^1	t_3^1	$\bar{x}_1^2$	x_2^2	$\bar{x}_3^2$	t_1^2	t_2^2	t_3^2
x_1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\bar{x}_1$	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_2$	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_3$	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_3^1	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_1^2$	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2^2	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_3^2$	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1^2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2^2	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_3^2	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5
Quantity of goods corresponding to cells in Block 1 for observation $\bar{x}_2$.

	x_1	$\bar{x}_1$	x_2	$\bar{x}_2$	x_3	$\bar{x}_3$	x_1^1	x_2^1	x_3^1	t_1^1	t_2^1	t_3^1	$\bar{x}_1^2$	x_2^2	$\bar{x}_3^2$	t_1^2	t_2^2	t_3^2
x_1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_1$	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_2$	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0
x_3	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_3$	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1^1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2^1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_3^1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1^1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2^1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_3^1	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_1^2$	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_2^2	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{x}_3^2$	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_1^2	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_2^2	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t_3^2	0	0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Specifying $q_{\chi_1^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 that are neither in row χ_1^ℓ nor in column χ_1^ℓ have quantity 0. The goods corresponding to cells $(\chi_1^\ell, \chi_1^\ell)$, $(\chi_1^\ell, \chi_2^\ell)$, $(\chi_1^\ell, \chi_3^\ell)$ and (χ_1^ℓ, t_3^ℓ) in row χ_1^ℓ have quantity 1. The good corresponding to cell $(\chi_2^\ell, \chi_1^\ell)$ gets a quantity of 4, that corresponding to cell $(\chi_3^\ell, \chi_1^\ell)$ receives a quantity of 6 while the good corresponding to cell (t_3^ℓ, χ_1^ℓ) has quantity 2. For the goods corresponding to the remaining cells in row χ_1^ℓ, their quantity is 0, while those corresponding to the remaining cells in column χ_1^ℓ have quantity Δ.

Specifying $q_{\chi_2^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 neither in row χ_2^ℓ nor in column χ_2^ℓ have quantity 0. The goods corresponding to cells $(\chi_2^\ell, \chi_2^\ell)$, $(\chi_2^\ell, \chi_1^\ell)$, $(\chi_2^\ell, \chi_3^\ell)$ and (χ_2^ℓ, t_1^ℓ) have quantity 1. The good corresponding to cell $(\chi_1^\ell, \chi_2^\ell)$ has a quantity of 6 and that corresponding to cell $(\chi_3^\ell, \chi_2^\ell)$ gets a quantity 4, while the good corresponding to cell (t_1^ℓ, χ_2^ℓ) has quantity 2. For the goods corresponding to the remaining cells in row χ_2^ℓ, their quantity is 0, while those corresponding to the remaining cells in column χ_2^ℓ have quantity Δ.

Specifying $q_{\chi_3^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 neither in row χ_3^ℓ nor in column χ_3^ℓ have quantity 0. The goods corresponding to cells $(\chi_3^\ell, \chi_3^\ell)$, $(\chi_3^\ell, \chi_1^\ell)$, $(\chi_3^\ell, \chi_2^\ell)$ and (χ_3^ℓ, t_2^ℓ) have quantity 1. In column χ_3^ℓ, the good corresponding to cell $(\chi_1^\ell, \chi_3^\ell)$ has a quantity of 4, that corresponding to cell $(\chi_2^\ell, \chi_3^\ell)$ has a quantity of 6 while the good corresponding to cell (t_2^ℓ, χ_3^ℓ) has quantity 2. For the goods corresponding to the remaining cells in row χ_3^ℓ, their quantity is 0, while those corresponding to the remaining cells in column χ_3^ℓ have quantity Δ.

Specifying $q_{t_1^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 that are neither in row t_1^ℓ nor in column t_1^ℓ have quantity 0. Since t_1^ℓ is associated to χ_1^ℓ which is either x_i or $\bar{x}_i$ for a given $i \in \{1, \dots, n\}$, we distinguish two cases.

If $\chi_1^\ell = x_i$, then the goods corresponding to cells (t_1^ℓ, t_1^ℓ), (t_1^ℓ, χ_2^ℓ) and $(t_1^\ell, \bar{x}_i)$ have quantity 1. In column t_1^ℓ, the good corresponding to cell (χ_2^ℓ, t_1^ℓ) has quantity 3, while the good corresponding to cell $(\bar{x}_i, t_1^\ell)$ has quantity $|\Gamma_{x_i}| + 1$. The goods corresponding to the remaining cells in row t_1^ℓ have quantity 0, while those corresponding to the remaining cells in column t_1^ℓ have quantity Δ.

If, on the other hand, $\chi_1^\ell = \bar{x}_i$ then the goods corresponding to cells (t_1^ℓ, t_1^ℓ), (t_1^ℓ, χ_2^ℓ) and (t_1^ℓ, x_i) have quantity 1. In column t_1^ℓ, the good corresponding to cell (χ_2^ℓ, t_1^ℓ) has quantity 3, while the good corresponding to cell (x_i, t_1^ℓ) has quantity $|\Gamma_{\bar{x}_i}| + 1$. The goods corresponding to the remaining cells in row t_1^ℓ have quantity 0, while those corresponding to the remaining cells in column t_1^ℓ have quantity Δ.

Specifying $q_{t_2^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 that are neither in row t_2^ℓ nor in column t_2^ℓ have quantity 0. Since t_2^ℓ is associated to χ_2^ℓ which is either x_j or $\bar{x}_j$ for a given $j \in \{1, \dots, n\}$, we distinguish two cases.

If $\chi_2^\ell = x_j$, then the goods corresponding to cells (t_2^ℓ, t_2^ℓ), (t_2^ℓ, χ_3^ℓ) and $(t_2^\ell, \bar{x}_j)$ have quantity 1. The good corresponding to cell (χ_3^ℓ, t_2^ℓ) has quantity 3, while the good corresponding to cell $(\bar{x}_j, t_2^\ell)$ has quantity $|\Gamma_{x_j}| + 1$. The goods corresponding to the remaining cells in row t_2^ℓ have quantity 0, while those corresponding to the remaining cells in column t_2^ℓ have quantity Δ.

Otherwise, if $\chi_2^\ell = \bar{x}_j$ then the goods corresponding to cells (t_2^ℓ, t_2^ℓ), (t_2^ℓ, χ_3^ℓ) and (t_2^ℓ, x_j) have quantity 1. In column t_2^ℓ, the good corresponding to cell (χ_3^ℓ, t_2^ℓ) has quantity 3, while the good corresponding to cell (x_j, t_2^ℓ) has quantity $|\Gamma_{\bar{x}_j}| + 1$. The goods corresponding to the remaining cells in row t_2^ℓ have quantity 0, while those corresponding to the remaining cells in column t_2^ℓ have quantity Δ.

Specifying $q_{t_3^\ell}$ for goods corresponding to cells in Block 1.

All goods corresponding to cells in Block 1 that are neither in row t_3^ℓ nor in column t_3^ℓ have quantity 0. Since t_3^ℓ is associated to χ_3^ℓ which is either x_k or $\bar{x}_k$ for a given $k \in \{1, \dots, n\}$, we distinguish two cases.

If $\chi_3^\ell = x_k$ then the goods corresponding to cells (t_3^ℓ, t_3^ℓ), (t_3^ℓ, χ_1^ℓ) and $(t_3^\ell, \bar{x}_k)$ have quantity 1. In column t_3^ℓ, the good corresponding to cell (χ_1^ℓ, t_3^ℓ) has quantity 3, while the good corresponding to cell $(\bar{x}_k, t_3^\ell)$ has quantity $|\Gamma_{x_k}| + 1$. The goods corresponding to the remaining cells in row t_3^ℓ have quantity 0, while those corresponding to the remaining cells in column t_3^ℓ have quantity Δ.

Otherwise, if $\chi_3^\ell = \bar{x}_k$, then the goods corresponding to cells (t_3^ℓ, t_3^ℓ), (t_3^ℓ, χ_1^ℓ) and (t_3^ℓ, x_k) have quantity 1. In column t_3^ℓ, the good corresponding to cell (χ_1^ℓ, t_3^ℓ) has quantity 3, while the good corresponding to cell (x_k, t_3^ℓ) has quantity $|\Gamma_{\bar{x}_k}| + 1$. The goods corresponding to the remaining cells in row t_3^ℓ have quantity 0, while those corresponding to the remaining cells in column t_3^ℓ have quantity Δ.

We now proceed with the quantities of the goods corresponding to cells in Block 2.

Specifying q_{x_i} for goods corresponding to cells in Block 2.

The goods corresponding to cells in Block 2 that have a non-ε price get the quantity $|\Gamma_{\bar{x}_i}| + 1$ while those with ε price get the value 0 as quantity.

As an illustration, consider the observation x_2 of the example. For that observation, all the goods corresponding to cells in Block 2 get the price ε. Therefore, all the goods corresponding to cells in Block 2 have quantity 0.

Specifying $q_{\bar{x}_i}$ for goods corresponding to cells in Block 2.

The goods corresponding to cells in Block 2 have quantity $|\Gamma_{x_i}| + 1$, if their price in that observation is different from ε; otherwise their quantity equals 0.

Specifying $q_{\chi_1^\ell}, q_{\chi_2^\ell}, q_{\chi_3^\ell}$ for goods corresponding to cells in Block 2.

For goods corresponding to cells in Block 2, the following holds: if the price of such a good in some observation is ε, then the quantity of that good for that observation is 0, otherwise the quantity is 3.

Specifying $q_{t_1^\ell}, q_{t_2^\ell}, q_{t_3^\ell}$ for goods corresponding to cells in Block 2.

For goods corresponding to cells in Block 2, the following holds: if the price of such a good in some observation is ε, then the quantity of that good for that observation is 0, otherwise the quantity is 2.

This completes the description of the quantity of each good in the bundle for every observation in $\mathbb{T}$. Thus, we have built the dataset S. Notice that this construction of S is done in polynomial time.

The second step of our proof identifies some characteristics of the dataset S constructed above; these are the inequalities and double-sum inequalities satisfied by the vectors of quantity and price of observations. Our goal is to compare for each pair s, t (respectively triple s, t_1, t_2) of observations the quantities $p_s q_s$ and $p_s q_t$ (respectively $p_s q_s$ and $p_s(q_{t_1} + q_{t_2})$).

Claim 1. *Given the dataset S defined above, we have the following inequalities.*

For each $i = 1, \dots, n$,

$$p_{x_i} q_{x_i} > p_{x_i} q_{\bar{x}_i}, \tag{1}$$
$$p_{\bar{x}_i} q_{\bar{x}_i} > p_{\bar{x}_i} q_{x_i}. \tag{2}$$

For each $\ell = 1, \ldots, m$,

$$p_{\chi_1^\ell} q_{\chi_1^\ell} > p_{\chi_1^\ell} q_{\chi_2^\ell}, \quad (3)$$

$$p_{\chi_1^\ell} q_{\chi_1^\ell} > p_{\chi_1^\ell} q_{\chi_3^\ell}, \quad (4)$$

$$p_{\chi_1^\ell} q_{\chi_1^\ell} > p_{\chi_1^\ell} q_{t_3^\ell}, \quad (5)$$

$$p_{\chi_2^\ell} q_{\chi_2^\ell} > p_{\chi_2^\ell} q_{\chi_1^\ell}, \quad (6)$$

$$p_{\chi_2^\ell} q_{\chi_2^\ell} > p_{\chi_2^\ell} q_{\chi_3^\ell}, \quad (7)$$

$$p_{\chi_2^\ell} q_{\chi_2^\ell} > p_{\chi_2^\ell} q_{t_1^\ell}, \quad (8)$$

$$p_{\chi_3^\ell} q_{\chi_3^\ell} > p_{\chi_3^\ell} q_{\chi_1^\ell}, \quad (9)$$

$$p_{\chi_3^\ell} q_{\chi_3^\ell} > p_{\chi_3^\ell} q_{\chi_2^\ell}, \quad (10)$$

$$p_{\chi_3^\ell} q_{\chi_3^\ell} > p_{\chi_3^\ell} q_{t_2^\ell}. \quad (11)$$

For each $\ell = 1, \ldots, m$, for each $i = 1, \ldots, n$ with $\chi_1^\ell = x_i$ or $\chi_1^\ell = \bar{x}_i$

$$p_{t_1^\ell} q_{t_1^\ell} > p_{t_1^\ell} q_{\chi_2^\ell}, \quad (12)$$

$$p_{t_1^\ell} q_{t_1^\ell} > \begin{cases} p_{t_1^\ell} q_{\bar{x}_i} & \text{if } \chi_1^\ell = x_i, \\ p_{t_1^\ell} q_{x_i} & \text{if } \chi_1^\ell = \bar{x}_i, \end{cases} \quad (13)$$

$$\left.\begin{array}{ll} p_{\bar{x}_i} q_{\bar{x}_i} > p_{\bar{x}_i} q_{t_1^\ell} & \text{if } \chi_1^\ell = x_i, \\ p_{x_i} q_{x_i} > p_{x_i} q_{t_1^\ell} & \text{if } \chi_1^\ell = \bar{x}_i. \end{array}\right\} \quad (14)$$

For each $\ell = 1, \ldots, m$, for each $j = 1, \ldots, n$ with $\chi_2^\ell = x_j$ or $\chi_2^\ell = \bar{x}_j$

$$p_{t_2^\ell} q_{t_2^\ell} > p_{t_2^\ell} q_{\chi_3^\ell}, \quad (15)$$

$$p_{t_2^\ell} q_{t_2^\ell} > \begin{cases} p_{t_2^\ell} q_{\bar{x}_j} & \text{if } \chi_2^\ell = x_j, \\ p_{t_2^\ell} q_{x_j} & \text{if } \chi_2^\ell = \bar{x}_j, \end{cases} \quad (16)$$

$$\left.\begin{array}{ll} p_{\bar{x}_j} q_{\bar{x}_j} > p_{\bar{x}_j} q_{t_2^\ell} & \text{if } \chi_2^\ell = x_j, \\ p_{x_j} q_{x_j} > p_{x_j} q_{t_2^\ell} & \text{if } \chi_2^\ell = \bar{x}_j. \end{array}\right\} \quad (17)$$

For each $\ell = 1, \ldots, m$, for each $k = 1, \ldots, n$ with $\chi_3^\ell = x_k$ or $\chi_3^\ell = \bar{x}_k$

$$p_{t_3^\ell} q_{t_3^\ell} > p_{t_3^\ell} q_{\chi_1^\ell}, \quad (18)$$

$$p_{t_3^\ell} q_{t_3^\ell} > \begin{cases} p_{t_3^\ell} q_{\bar{x}_k} & \text{if } \chi_3^\ell = x_k, \\ p_{t_3^\ell} q_{x_k} & \text{if } \chi_3^\ell = \bar{x}_k, \end{cases} \quad (19)$$

$$\left.\begin{array}{ll} p_{\bar{x}_k} q_{\bar{x}_k} > p_{\bar{x}_k} q_{t_3^\ell} & \text{if } \chi_3^\ell = x_k, \\ p_{x_k} q_{x_k} > p_{x_k} q_{t_3^\ell} & \text{if } \chi_3^\ell = \bar{x}_k. \end{array}\right\} \quad (20)$$

For all other pair s, t of distinct observations in $\mathbb{T}$

$$p_s q_s < p_s q_t. \quad (21)$$

Claim 2. *Given the dataset S defined above, the following double-sum inequalities hold.*

For each $\ell = 1, \ldots, m$,

$$p_{\chi_1^\ell} q_{\chi_1^\ell} > p_{\chi_1^\ell}(q_{\chi_3^\ell} + q_{t_3^\ell}), \quad (22)$$

$$p_{\chi_2^\ell} q_{\chi_2^\ell} > p_{\chi_2^\ell}(q_{\chi_1^\ell} + q_{t_1^\ell}), \quad (23)$$

$$p_{\chi_3^\ell} q_{\chi_3^\ell} > p_{\chi_3^\ell}(q_{\chi_2^\ell} + q_{t_2^\ell}). \quad (24)$$

For each $\ell = 1, \ldots, m$, for each $i = 1, \ldots, n$ with $\chi_1^\ell = x_i$ or $\chi_1^\ell = \bar{x}_i$

$$p_{t_1^\ell} q_{t_1^\ell} > \begin{cases} p_{t_1^\ell}(q_{\chi_2^\ell} + q_{\bar{x}_i}) & \text{if } \chi_1^\ell = x_i, \\ p_{t_1^\ell}(q_{\chi_2^\ell} + q_{x_i}) & \text{if } \chi_1^\ell = \bar{x}_i, \end{cases} \quad (25)$$

$$\left.\begin{array}{ll} p_{\bar{x}_i} q_{\bar{x}_i} > p_{\bar{x}_i}(q_{x_i} + q_{t_1^\ell}) & \text{if } \chi_1^\ell = x_i, \\ p_{x_i} q_{x_i} > p_{x_i}(q_{\bar{x}_i} + q_{t_1^\ell}) & \text{if } \chi_1^\ell = \bar{x}_i. \end{array}\right\} \quad (26)$$

For each $\ell = 1, \ldots, m$, for each $j = 1, \ldots, n$ with $\chi_2^\ell = x_j$ or $\chi_2^\ell = \bar{x}_j$

$$p_{t_2^\ell} q_{t_2^\ell} > \begin{cases} p_{t_2^\ell}(q_{\chi_3^\ell} + q_{\bar{x}_j}) & \text{if } \chi_2^\ell = x_j, \\ p_{t_2^\ell}(q_{\chi_3^\ell} + q_{x_j}) & \text{if } \chi_2^\ell = \bar{x}_j, \end{cases} \quad (27)$$

$$\left.\begin{array}{ll} p_{\bar{x}_j} q_{\bar{x}_j} > p_{\bar{x}_j}(q_{x_j} + q_{t_2^\ell}) & \text{if } \chi_2^\ell = x_j, \\ p_{x_j} q_{x_j} > p_{x_j}(q_{\bar{x}_j} + q_{t_2^\ell}) & \text{if } \chi_2^\ell = \bar{x}_j. \end{array}\right\} \quad (28)$$

For each $\ell = 1, \ldots, m$, for each $k = 1, \ldots, n$ with $\chi_3^\ell = x_k$ or $\chi_3^\ell = \bar{x}_k$

$$p_{t_3^\ell} q_{t_3^\ell} > \begin{cases} p_{t_3^\ell}(q_{\chi_1^\ell} + q_{\bar{x}_k}) & \text{if } \chi_3^\ell = x_k, \\ p_{t_3^\ell}(q_{\chi_1^\ell} + q_{x_k}) & \text{if } \chi_3^\ell = \bar{x}_k, \end{cases} \quad (29)$$

$$\left.\begin{array}{ll} p_{\bar{x}_k} q_{\bar{x}_k} > p_{\bar{x}_k}(q_{x_k} + q_{t_3^\ell}) & \text{if } \chi_3^\ell = x_k, \\ p_{x_k} q_{x_k} > p_{x_k}(q_{\bar{x}_k} + q_{t_3^\ell}) & \text{if } \chi_3^\ell = \bar{x}_k. \end{array}\right\} \quad (30)$$

For each $i = 1, \ldots, n$ and for each $\ell, \ell' \in \Gamma_{\bar{x}_i}$ with r and r' being the position of $\bar{x}_i$ in the clause C_ℓ and $C_{\ell'}$ respectively,

$$p_{x_i} q_{x_i} > p_{x_i}(q_{t_r^\ell} + q_{t_{r'}^{\ell'}}). \quad (31)$$

For each $i = 1, \ldots, n$ and for each $\ell, \ell' \in \Gamma_{x_i}$ with r and r' being the position of x_i in the clause C_ℓ and $C_{\ell'}$ respectively,

$$p_{\bar{x}_i} q_{\bar{x}_i} > p_{\bar{x}_i}(q_{t_r^\ell} + q_{t_{r'}^{\ell'}}). \quad (32)$$

There are no double-sum inequalities other than those mentioned above.

The proofs of Claims 1 and 2 are given in the Appendices B and C.

In the last step of our proof, we prove that the dataset S obtained by the above reduction satisfies CARP if and only if the instance of the Not-All-Equal-3Sat problem is a YES-instance. The goal here is to prove that the instance of CARP built from the arbitrary instance of the Not-All-Equal-3Sat problem is at least as hard as that instance of the Not-All-Equal-3Sat problem. This proof strongly relies on Claims 1 and 2.

On the one hand, suppose that S satisfies CARP. Thus there exist sets (hypothetical relations) H_0^1 and H_0^2 that satisfy *Rules* 1–5. The following is true for H_0^1 and H_0^2.

Lemma 1. *If the dataset S satisfies CARP, then there are no two distinct observations s and t satisfying $p_s q_s \geq p_s q_t$ such that $(q_s, q_t) \in H_0^1$ and $(q_s, q_t) \in H_0^2$.*

Proof. From the inequalities listed in Claim 1, we observe that for a pair of distinct observations $s, t \in \mathbb{T}$, if $p_s q_s \geq p_s q_t$ then $p_s q_s > p_s q_t$ and $p_t q_t > p_t q_s$. Next, we argue by contradiction: if $(q_s, q_t) \in H_0^1$ and $(q_s, q_t) \in H_0^2$ then $(q_s, q_t) \in H^1$ and $(q_s, q_t) \in H^2$. This, however, contradicts *Rule* 4. Therefore, Lemma 1 holds. □

We now build a truth assignment for the instance of Not-All-Equal-3Sat and show that it is a yes-instance. For each variable $x_i \in X$ we set $x_i = 1$ if $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$; otherwise $x_i = 0$. Thus, the value of each x_i is well defined. In fact, using (1) and *Rule 1* we conclude that for each i, $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$ or $(q_{x_i}, q_{\bar{x}_i}) \in H_0^2$. Since by construction, $x_i = 1$ corresponds to the case $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$, it follows that $x_i = 0$ corresponds to the case $(q_{x_i}, q_{\bar{x}_i}) \in H_0^2$.

We now prove that each clause in C has at least one true literal and at least one false literal. We argue by contradiction. Suppose that there exists a clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell)$ ($\ell \in \{1, \ldots, m\}$) in C which is such that either $\chi_1^\ell = \chi_2^\ell = \chi_3^\ell = 1$ or $\chi_1^\ell =$

$\chi_2^\ell = \chi_3^\ell = 0$. Without loss of generality, let us assume that $\chi_1^\ell = \chi_2^\ell = \chi_3^\ell = 1$. We are going to investigate each literal in C_ℓ individually. The first literal χ_1^ℓ is either x_i or $\bar{x}_i$. We will argue that in both cases, we have $(q_{t_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$.

Indeed, if $\chi_1^\ell = x_i$ then $x_i = 1$ implies that $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$ from the assignment of values to variables. The double-sum inequality (26) for the clause C_ℓ is $p_{\bar{x}_i}q_{\bar{x}_i} > p_{\bar{x}_i}(q_{x_i} + q_{t_1^\ell})$. Since $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$, *Rule 3* implies that $(q_{\bar{x}_i}, q_{t_1^\ell}) \in H_0^2$. Using the double-sum inequality $p_{t_1^\ell}q_{t_1^\ell} > p_{t_1^\ell}(q_{\chi_2^\ell} + q_{\bar{x}_i})$ given by (25) and the fact that $(q_{\bar{x}_i}, q_{t_1^\ell}) \in H_0^2$, *Rule 3* leads to $(q_{t_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$. On the other hand, if $\chi_1^\ell = \bar{x}_i$ then $\bar{x}_i = 1$ implies that $x_i = 0$ and $(q_{x_i}, q_{\bar{x}_i}) \in H_0^2$. Using *Rule 2* and (2) we obtain $(q_{\bar{x}_i}, q_{x_i}) \in H_0^1$. The double-sum inequality (26) is $p_{x_i}q_{x_i} > p_{x_i}(q_{\bar{x}_i} + q_{t_1^\ell})$ and $(q_{\bar{x}_i}, q_{x_i}) \in H_0^1$. Thus *Rule 3* implies that $(q_{x_i}, q_{t_1^\ell}) \in H_0^2$. The inequality (25) is $p_{t_1^\ell}q_{t_1^\ell} > p_{t_1^\ell}(q_{\chi_2^\ell} + q_{x_i})$ and $(q_{x_i}, q_{t_1^\ell}) \in H_0^2$; therefore $(q_{t_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$ from *Rule 3*.

Since $(q_{t_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$ we use the double-sum inequality $p_{\chi_2^\ell}q_{\chi_2^\ell} > p_{\chi_2^\ell}(q_{\chi_1^\ell} + q_{t_1^\ell})$, (given by (23)) and *Rule 3* to obtain that $(q_{\chi_2^\ell}, q_{\chi_1^\ell}) \in H_0^2$. Finally, using *Rule 2* and (8) we have $(q_{\chi_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$.

We conclude that whether the literal χ_1^ℓ is x_i or $\bar{x}_i$, as long as its value equals 1 we have $(q_{\chi_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$. Notice that, in case $\chi_1^\ell = 0$ we can conclude that $(q_{\chi_1^\ell}, q_{\chi_2^\ell}) \in H_0^2$.

By applying a similar reasoning to the second literal χ_2^ℓ, we obtain $(q_{\chi_2^\ell}, q_{\chi_3^\ell}) \in H_0^1$, while the application of that reasoning to the third literal χ_3^ℓ leads to $(q_{\chi_3^\ell}, q_{\chi_1^\ell}) \in H_0^1$. We obtain $(q_{\chi_1^\ell}, q_{\chi_2^\ell}) \in H_0^1$, $(q_{\chi_2^\ell}, q_{\chi_3^\ell}) \in H_0^1$ and $(q_{\chi_3^\ell}, q_{\chi_1^\ell}) \in H_0^1$. Thus $(q_{\chi_1^\ell}, q_{\chi_2^\ell})$, $(q_{\chi_2^\ell}, q_{\chi_3^\ell})$, $(q_{\chi_3^\ell}, q_{\chi_1^\ell}) \in H^1$, which imply from *Rule 2*, (4), (6) and (10) that $(q_{\chi_1^\ell}, q_{\chi_2^\ell})$, $(q_{\chi_2^\ell}, q_{\chi_3^\ell})$, $(q_{\chi_3^\ell}, q_{\chi_1^\ell}) \in H_0^2$. Thus $(q_{\chi_1^\ell}, q_{\chi_2^\ell}) \in H_0^1 \cap H_0^2$ and $\chi_2^\ell \neq \chi_1^\ell$. This contradicts Lemma 1. This concludes the proof that if the dataset S satisfies CARP then the instance of Not-All-Equal-3Sat is a yes-instance.

On the other hand, suppose that there is a truth assignment for X which is such that each clause in C has at least one true literal and at least one false literal. Consider H_0^1 and H_0^2 defined as follows. For each variable $x_i \in X$, if $x_i = 1$ then $(q_{x_i}, q_{\bar{x}_i}) \in H_0^1$ and $(q_{\bar{x}_i}, q_{x_i}) \in H_0^2$. Otherwise, if $x_i = 0$ then $(q_{\bar{x}_i}, q_{x_i}) \in H_0^1$ and $(q_{x_i}, q_{\bar{x}_i}) \in H_0^2$. This ensures that for each pair of observations (s, t) occurring in inequalities (1) or (2) the corresponding bundle pair (q_s, q_t) is either in H_0^1 or in H_0^2. We now deal with pairs of observations occurring in inequalities (3)–(20). We will specify for each ordered pair of observations occurring in each of these inequalities whether the corresponding bundle pair is in H_0^1 or in H_0^2. For every clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell)$ in C, we consider each literal in C_ℓ in turn. The construction of H_0^1 and H_0^2 for a given clause C_ℓ is given in Table 6.

Table 6 displays two forms of symmetry. First, at the level of a literal χ_i^ℓ, $i = 1, 2, 3$ we observe that the set H_0^1 when $\chi_i^\ell = 1$ equals the set H_0^2 when $\chi_i^\ell = 0$. Second, when substituting $\bar{x}_i$ $(\bar{x}_j, \bar{x}_k)$ for x_i (x_j, x_k), and x_i (x_j, x_k) for $\bar{x}_i$ $(\bar{x}_j, \bar{x}_k)$, the set H_0^1 (respectively H_0^2) when $\chi_1^\ell = x_i$ $(\chi_2^\ell = x_j, \chi_3^\ell = x_k)$ becomes the set H_0^2 (respectively H_0^1) when $\chi_1^\ell = \bar{x}_i$ $(\chi_2^\ell = \bar{x}_j, \chi_3^\ell = \bar{x}_k)$.

To complete the definition of H_0^1 and H_0^2, we set $(q_s, q_s) \in H_0^1 \cap H_0^2$ for every $s \in \mathbb{T}$.

Remark 1. Notice that there is no pair of distinct observations (s, t) such that $p_sq_s \geq p_sq_t$ and $(q_s, q_t) \in H_0^1 \cap H_0^2$.

We next prove two properties of the sets H_0^1, H_0^2, H^1 and H^2 described above.

Property 1. *For any pair of observations (s, t), if $(q_s, q_t) \in H^i$ and $p_sq_s \geq p_sq_t$ then $(q_s, q_t) \in H_0^i$, for $i = 1, 2$.*

Proof. Without loss of generality, suppose that $(q_s, q_t) \in H^1$ with $p_sq_s \geq p_sq_t$. We argue by contradiction; suppose that $(q_s, q_t) \notin H_0^1$. Then by construction, $(q_s, q_t) \in H_0^2$. Since $(q_s, q_t) \in H_0^2$, we have, by construction of H_0^1 and H_0^2, that $(q_t, q_s) \in H_0^1$. Further, since $(q_s, q_t) \in H^1$ there exists a sequence (non-empty, because of Remark 1) of observations $u, v, \ldots, w$ such that $(q_s, q_u), (q_u, q_v), \ldots, (q_w, q_t)$ are in H_0^1. By construction of H_0^1 and H_0^2, however, this implies that $(q_t, q_w), \ldots, (q_v, q_u), (q_u, q_s) \in H_0^2$. Together with the fact that $(q_s, q_t) \in H_0^2$ and $(q_t, q_s) \in H_0^1$, we get $(q_s, q_u), (q_u, q_v), \ldots, (q_w, q_t), (q_t, q_s) \in H_0^1$ and $(q_s, q_t), (q_t, q_w), \ldots, (q_v, q_u), (q_u, q_s) \in H_0^2$. In other words, for every observation $a \in \{s, t, u, \ldots, w\}$ there exist two observations b and c in $\{s, t, u, \ldots, w\}$ such that $(q_a, q_b) \in H_0^1$ and $(q_a, q_c) \in H_0^2$. It follows from the construction (see Table 6) that this can happen only if $a \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$ for a given $\ell = 1, \ldots, m$. Therefore, $s, t, u, \ldots, w \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$ for a given $\ell = 1, \ldots, m$. The latter result implies that the length of the sequence $u, v, \ldots, w$ is one (suppose that the sequence contains only u) and we have (q_s, q_u) and (q_u, q_t) in H_0^1. If, in addition $(q_t, q_s) \in H_0^1$, then we have $(q_{\chi_1^\ell}, q_{\chi_2^\ell})$, $(q_{\chi_2^\ell}, q_{\chi_3^\ell})$ and $(q_{\chi_3^\ell}, q_{\chi_1^\ell})$ in H_0^1, which is only possible if the variables χ_1^ℓ, χ_2^ℓ and χ_3^ℓ are assigned the same value. Since these three variables are in the same clause C_ℓ, this contradicts the fact that we have a truth assignment that is a solution to the Not-All-Equal-3Sat instance. □

Property 2. *For any triple of observations (s, t_1, t_2) satisfying $p_sq_s \geq p_s(q_{t_1} + q_{t_2})$, the pair of bundles (q_s, q_{t_1}) and (q_s, q_{t_2}) are in the same set H_0^i for $i = 1, 2$.*

Proof. It is not difficult to check this result from Table 6. □

We now prove that the hypothetical relations H_0^1 and H_0^2 defined above satisfy *Rules* 1–5.

Rule 1: This rule is satisfied because on the one hand, a pair of distinct observations s, t in $\mathbb{T}$ occurring in $p_sq_s \geq p_sq_t$ is identified by one of the inequalities (1)–(20) and therefore is by construction either in H_0^1 or in H_0^2. On the other hand, $(q_s, q_s) \in H_0^1 \cap H_0^2$ by construction.

Rule 2: Suppose that $p_sq_s \geq p_sq_t$ and $(q_t, q_s) \in H^1$. We know by construction that $p_sq_s \geq p_sq_t$ implies $p_tq_t \geq p_tq_s$ and from Property 1 we have $(q_t, q_s) \in H_0^1$; which implies $(q_s, q_t) \in H_0^2$.

Rule 3: Suppose that $p_sq_s \geq p_s(q_{t_1} + q_{t_2})$ and $(q_{t_1}, q_s) \in H^1$. Property 1 implies that $(q_{t_1}, q_s) \in H_0^1$ and Property 2 implies that (q_s, q_{t_1}) and (q_s, q_{t_2}) are in the same set. Since $(q_{t_1}, q_s) \in H_0^1$, we conclude that $(q_s, q_{t_2}) \in H_0^2$.

Rule 4: This rule follows from the fact that $H_0^1 \cap H_0^2$ contains only (q_s, q_s) where s is a given observation in $\mathbb{T}$. Thus, for two distinct observations s and t with $p_sq_s > p_sq_t$, $(q_s, q_t) \notin H_0^1 \cap H_0^2$.

Rule 5: Suppose that there exist $s, t_1, t_2 \in \mathbb{T}$ such that $p_sq_s > p_s(q_{t_1} + q_{t_2})$ and $(q_{t_1}, q_s) \in H^1$. Then $(q_{t_1}, q_s) \in H_0^1$ from Property 1 and $(q_s, q_{t_2}) \in H_0^2$ from Property 2. Therefore $(q_{t_2}, q_s) \notin H_0^2$ because of Lemma 1.

This concludes the proof that if the instance of the Not-All-Equal-3Sat is a yes-instance then the dataset S satisfies CARP.

Table 6
Construction of H_0^1 and H_0^2 for a given clause C_ℓ.

	$\chi_1^\ell = x_i$				$\chi_1^\ell = \bar{x}_i$			
	$x_i = 1$	Ineq.	$x_i = 0$	Ineq.	$x_i = 1$	Ineq.	$x_i = 0$	Ineq.
H_0^1	$(q_{x_i}, q_{\bar{x}_i})$	(1)	$(q_{\bar{x}_i}, q_{x_i})$	(2)	$(q_{x_i}, q_{\bar{x}_i})$	(1)	$(q_{\bar{x}_i}, q_{x_i})$	(2)
	$(q_{t_1^\ell}, q_{\bar{x}_i})$	(13)	$(q_{\bar{x}_i}, q_{t_1^\ell})$	(14)	$(q_{x_i}, q_{t_1^\ell})$	(14)	$(q_{t_1^\ell}, q_{x_i})$	(13)
	$(q_{t_1^\ell}, q_{\chi_2^\ell})$	(12)	$(q_{\chi_2^\ell}, q_{t_1^\ell})$	(8)	$(q_{\chi_2^\ell}, q_{t_1^\ell})$	(8)	$(q_{t_1^\ell}, q_{\chi_2^\ell})$	(12)
	$(q_{\chi_1^\ell}, q_{\chi_2^\ell})$	(3)	$(q_{\chi_2^\ell}, q_{\chi_1^\ell})$	(6)	$(q_{\chi_2^\ell}, q_{\chi_1^\ell})$	(6)	$(q_{\chi_1^\ell}, q_{\chi_2^\ell})$	(3)
H_0^2	$(q_{\bar{x}_i}, q_{x_i})$	(2)	$(q_{x_i}, q_{\bar{x}_i})$	(1)	$(q_{\bar{x}_i}, q_{x_i})$	(2)	$(q_{x_i}, q_{\bar{x}_i})$	(1)
	$(q_{\bar{x}_i}, q_{t_1^\ell})$	(14)	$(q_{t_1^\ell}, q_{\bar{x}_i})$	(13)	$(q_{t_1^\ell}, q_{x_i})$	(13)	$(q_{x_i}, q_{t_1^\ell})$	(14)
	$(q_{\chi_2^\ell}, q_{t_1^\ell})$	(8)	$(q_{t_1^\ell}, q_{\chi_2^\ell})$	(12)	$(q_{t_1^\ell}, q_{\chi_2^\ell})$	(12)	$(q_{\chi_2^\ell}, q_{t_1^\ell})$	(8)
	$(q_{\chi_2^\ell}, q_{\chi_1^\ell})$	(6)	$(q_{\chi_1^\ell}, q_{\chi_2^\ell})$	(3)	$(q_{\chi_1^\ell}, q_{\chi_2^\ell})$	(3)	$(q_{\chi_2^\ell}, q_{\chi_1^\ell})$	(6)
	$\chi_2^\ell = x_j$				$\chi_2^\ell = \bar{x}_j$			
	$x_j = 1$	Ineq.	$x_j = 0$	Ineq.	$x_j = 1$	Ineq.	$x_j = 0$	Ineq.
H_0^1	$(q_{x_j}, q_{\bar{x}_j})$	(1)	$(q_{\bar{x}_j}, q_{x_j})$	(2)	$(q_{x_j}, q_{\bar{x}_j})$	(1)	$(q_{\bar{x}_j}, q_{x_j})$	(2)
	$(q_{t_2^\ell}, q_{\bar{x}_j})$	(16)	$(q_{\bar{x}_j}, q_{t_2^\ell})$	(17)	$(q_{x_j}, q_{t_2^\ell})$	(17)	$(q_{t_2^\ell}, q_{x_j})$	(16)
	$(q_{t_2^\ell}, q_{\chi_3^\ell})$	(15)	$(q_{\chi_3^\ell}, q_{t_2^\ell})$	(11)	$(q_{\chi_3^\ell}, q_{t_2^\ell})$	(11)	$(q_{t_2^\ell}, q_{\chi_3^\ell})$	(15)
	$(q_{\chi_2^\ell}, q_{\chi_3^\ell})$	(7)	$(q_{\chi_3^\ell}, q_{\chi_2^\ell})$	(10)	$(q_{\chi_3^\ell}, q_{\chi_2^\ell})$	(10)	$(q_{\chi_2^\ell}, q_{\chi_3^\ell})$	(7)
H_0^2	$(q_{\bar{x}_j}, q_{x_j})$	(2)	$(q_{x_j}, q_{\bar{x}_j})$	(1)	$(q_{\bar{x}_j}, q_{x_j})$	(2)	$(q_{x_j}, q_{\bar{x}_j})$	(1)
	$(q_{\bar{x}_j}, q_{t_2^\ell})$	(17)	$(q_{t_2^\ell}, q_{\bar{x}_j})$	(16)	$(q_{t_2^\ell}, q_{x_j})$	(16)	$(q_{x_j}, q_{t_2^\ell})$	(17)
	$(q_{\chi_3^\ell}, q_{t_2^\ell})$	(11)	$(q_{t_2^\ell}, q_{\chi_3^\ell})$	(15)	$(q_{t_2^\ell}, q_{\chi_3^\ell})$	(15)	$(q_{\chi_3^\ell}, q_{t_2^\ell})$	(11)
	$(q_{\chi_3^\ell}, q_{\chi_2^\ell})$	(10)	$(q_{\chi_2^\ell}, q_{\chi_3^\ell})$	(7)	$(q_{\chi_2^\ell}, q_{\chi_3^\ell})$	(7)	$(q_{\chi_3^\ell}, q_{\chi_2^\ell})$	(10)
	$\chi_3^\ell = x_k$				$\chi_3^\ell = \bar{x}_k$			
	$x_k = 1$	Ineq.	$x_k = 0$	Ineq.	$x_k = 1$	Ineq.	$x_k = 0$	Ineq.
H_0^1	$(q_{x_k}, q_{\bar{x}_k})$	(1)	$(q_{\bar{x}_k}, q_{x_k})$	(2)	$(q_{x_k}, q_{\bar{x}_k})$	(1)	$(q_{\bar{x}_k}, q_{x_k})$	(2)
	$(q_{t_3^\ell}, q_{\bar{x}_k})$	(19)	$(q_{\bar{x}_k}, q_{t_3^\ell})$	(20)	$(q_{x_k}, q_{t_3^\ell})$	(20)	$(q_{t_3^\ell}, q_{x_k})$	(19)
	$(q_{t_3^\ell}, q_{\chi_1^\ell})$	(18)	$(q_{\chi_1^\ell}, q_{t_3^\ell})$	(5)	$(q_{\chi_1^\ell}, q_{t_3^\ell})$	(5)	$(q_{t_3^\ell}, q_{\chi_1^\ell})$	(18)
	$(q_{\chi_3^\ell}, q_{\chi_1^\ell})$	(9)	$(q_{\chi_1^\ell}, q_{\chi_3^\ell})$	(4)	$(q_{\chi_1^\ell}, q_{\chi_3^\ell})$	(4)	$(q_{\chi_3^\ell}, q_{\chi_1^\ell})$	(9)
H_0^2	$(q_{\bar{x}_k}, q_{x_k})$	(2)	$(q_{x_k}, q_{\bar{x}_k})$	(1)	$(q_{\bar{x}_k}, q_{x_k})$	(2)	$(q_{x_k}, q_{\bar{x}_k})$	(1)
	$(q_{\bar{x}_k}, q_{t_3^\ell})$	(20)	$(q_{t_3^\ell}, q_{\bar{x}_k})$	(19)	$(q_{t_3^\ell}, q_{x_k})$	(19)	$(q_{x_k}, q_{t_3^\ell})$	(20)
	$(q_{\chi_1^\ell}, q_{t_3^\ell})$	(5)	$(q_{t_3^\ell}, q_{\chi_1^\ell})$	(18)	$(q_{t_3^\ell}, q_{\chi_1^\ell})$	(18)	$(q_{\chi_1^\ell}, q_{t_3^\ell})$	(5)
	$(q_{\chi_1^\ell}, q_{\chi_3^\ell})$	(4)	$(q_{\chi_3^\ell}, q_{\chi_1^\ell})$	(9)	$(q_{\chi_3^\ell}, q_{\chi_1^\ell})$	(9)	$(q_{\chi_1^\ell}, q_{\chi_3^\ell})$	(4)

4. Conclusion

This text proves that the problem of testing the Collective Axiom of Revealed Preference (CARP) is NP-complete even for two-member household. This result justifies the enumerative approaches that are used in Cherchye et al. (2008) and the heuristic approaches used in Talla Nobibon et al. (forthcoming) to test a given dataset for CARP.

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Appendix A. Scalar product of observations in $\mathbb{T}$

In this section, we derive the value of the scalar product $p_s q_t$ for each pair of observations s and t in $\mathbb{T}$.

In what follows, we first specify the quantity $p_s q_t$ for every pair of observations s and t in $\mathbb{T}$. The symbol $\cong$ is used to mean that the value reported of $p_s q_t$ is the limit when ϵ tends to 0 of the exact value. We consider five cases.

Case 1. Both observations are variable observations; that is $s, t \in \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$.

Below, we distinguish eight types of combinations as follows:

How to compute $p_{x_i} q_{x_i}$ when $s = t = x_i$ with $i = 1, \ldots, n$.
Notice that the scalar product $p_{x_i} q_{x_i}$ is not affected by goods corresponding to cells with quantity zero. Further, as we take the limit when ε tends to zero, only goods corresponding to cells in Block 1 and in Block 2 with price different from ε are considered. In Block 1, this restriction allows to consider only goods corresponding to cells in row x_i. The quantity $p_{x_i} q_{x_i}$ contains a part coming from the good corresponding to cell (x_i, x_i) in Block 1. This accounts for $2 \times 1 = 1$ in $p_{x_i} q_{x_i}$ since the price is 2 and the quantity is 1. Looking at the vector of quantity, the good corresponding to cell $(x_i, \bar{x}_i)$ has a quantity of one and contributes for $1 \times 1 = 1$ in $p_{x_i} q_{x_i}$. Moreover, we know that for every clause C_ℓ with $\ell \in \Gamma_{\bar{x}_i}$, the good corresponding to cell (x_i, t_r^ℓ) in row x_i gets the value one (here, $r \in \{1, 2, 3\}$ is the position of $\bar{x}_i$ in the clause C_ℓ). Thus each such good adds the value of one to $p_{x_i} q_{x_i}$ and there are $|\Gamma_{\bar{x}_i}|$ such goods. There are no additional value coming from the goods corresponding to the remaining cells in Block 1. In total, goods corresponding to cells in Block 1 contribute for $2 + 1 + |\Gamma_{\bar{x}_i}|$ in $p_{x_i} q_{x_i}$. For goods corresponding to cells in Block 2, we know by construction that there are $2 \times |\Gamma_{\bar{x}_i}|$ goods with price $\frac{1}{2 \times |\Gamma_{\bar{x}_i}|}$ and the quantity of $|\Gamma_{\bar{x}_i}| + 1$. Therefore, if $\Gamma_{\bar{x}_i} \neq \emptyset$ then the goods corresponding to cells in Block 2 contribute for $2 \times |\Gamma_{\bar{x}_i}| \left(\frac{1}{2 \times |\Gamma_{\bar{x}_i}|} \times (|\Gamma_{\bar{x}_i}| + 1) \right) = |\Gamma_{\bar{x}_i}| + 1$. Notice that if $\Gamma_{\bar{x}_i} = \emptyset$ then that contribution is zero. Putting together the contribution of goods corresponding to cells in Block 1 and in Block 2, we obtain $p_{x_i} q_{x_i} \cong 2 + 1 + |\Gamma_{\bar{x}_i}| + |\Gamma_{\bar{x}_i}| + 1 = 2|\Gamma_{\bar{x}_i}| + 4$ if $\Gamma_{\bar{x}_i} \neq \emptyset$ and $p_{x_i} q_{x_i} \cong 2 + 1$ if $\Gamma_{\bar{x}_i} = \emptyset$. Therefore,

$$p_{x_i} q_{x_i} \cong \begin{cases} 3 & \text{if } \Gamma_{\bar{x}_i} = \emptyset \\ 2|\Gamma_{\bar{x}_i}| + 4 & \text{if } \Gamma_{\bar{x}_i} \neq \emptyset. \end{cases} \quad (33)$$

How to compute $p_{\bar{x}_i} q_{\bar{x}_i}$ when $s = t = \bar{x}_i$ with $i = 1, \ldots, n$.

Following the procedure above, the scalar product

$$p_{\bar{x}_i}q_{\bar{x}_i} \cong \begin{cases} 3 & \text{if } \Gamma_{x_i} = \emptyset \\ 2|\Gamma_{x_i}| + 4 & \text{if } \Gamma_{x_i} \neq \emptyset. \end{cases} \quad (34)$$

How to compute $p_{x_i}q_{\bar{x}_i}$ when $s = x_i$ and $t = \bar{x}_i$ with $i = 1, \ldots, n$.
To compute the scalar product $p_{x_i}q_{\bar{x}_i}$, observe that goods corresponding to cells in Block 2 do not affect that scalar product. In fact, for the observation x_i, the only goods in Block 2 with price different from ε corresponding to cells either in row $\bar{x}_i$ or in column $\bar{x}_i$ while the goods of observation $\bar{x}_i$ in Block 2 with non-zero quantity corresponding to cells either in row x_i or in column x_i. Therefore, the scalar product $p_{x_i}q_{\bar{x}_i}$ is based on goods corresponding to cells in Block 1. In that block, only goods corresponding to cells in row x_i are interesting as they have a non-ε price. However, for observation $\bar{x}_i$ the only good in row x_i of Block 1 with non-zero quantity corresponding to cell $(x_i, \bar{x}_i)$ with quantity $|\Gamma_{\bar{x}_i}| + 1$. This implies that

$$p_{x_i}q_{\bar{x}_i} \cong |\Gamma_{\bar{x}_i}| + 1. \quad (35)$$

How to compute $p_{\bar{x}_i}q_{x_i}$ when $s = \bar{x}_i$ and $t = x_i$ with $i = 1, \ldots, n$.
An analysis following the reasoning used above leads to

$$p_{\bar{x}_i}q_{x_i} \cong |\Gamma_{x_i}| + 1. \quad (36)$$

How to compute $p_{x_i}q_{x_j}$ when $s = x_i$ and $t = x_j$ with $i, j \in \{1, \ldots, n\}, i \neq j$.
We are not going to compute $p_{x_i}q_{x_j}$ but we will rather provide a lower bound to $p_{x_i}q_{x_j}$. Notice that the scalar product $p_{x_i}q_{x_j}$ is at least as large as the contribution of good corresponding to cell (x_i, x_j) in Block 1. The latter good contributes $1 \times \Delta = \Delta$ to $p_{x_i}q_{x_j}$ because the cell (x_i, x_j) is in the column x_j and in that column, the observation x_j is such that only goods corresponding to cells (x_j, x_j), $(\bar{x}_j, x_j)$ and (t_r^ℓ, x_j) where the clause C_ℓ contains $\bar{x}_j$ (r being the position of $\bar{x}_j$ in C_ℓ), have quantity different from Δ. Therefore

$$p_{x_i}q_{x_j} \geq \Delta. \quad (37)$$

How to compute $p_{\bar{x}_i}q_{x_j}$ when $s = \bar{x}_i$ and $t = x_j$ with $i, j \in \{1, \ldots, n\}, i \neq j$.
Similarly, $p_{\bar{x}_i}q_{x_j}$ is greater than or equal to Δ using the same reasoning as above. Thus,

$$p_{\bar{x}_i}q_{x_j} \geq \Delta. \quad (38)$$

How to compute $p_{x_i}q_{\bar{x}_j}$ when $s = x_i$ and $t = \bar{x}_j$ with $i, j \in \{1, \ldots, n\}, i \neq j$.

The quantity $p_{x_i}q_{\bar{x}_j}$ is at least as large as the contribution of the good corresponding to cell $(x_i, \bar{x}_j)$ in Block 1. That contribution is $1 \times \Delta = \Delta$. Therefore

$$p_{x_i}q_{\bar{x}_j} \geq \Delta. \quad (39)$$

How to compute $p_{\bar{x}_i}q_{\bar{x}_j}$ when $s = \bar{x}_i$ and $t = \bar{x}_j$ with $i, j \in \{1, \ldots, n\}, i \neq j$.
The quantity $p_{\bar{x}_i}q_{\bar{x}_j}$ is at least as large as the contribution of the good corresponding to cell $(\bar{x}_i, \bar{x}_j)$ in Block 1. However, that good contributes for $1 \times \Delta = \Delta$. Therefore

$$p_{\bar{x}_i}q_{\bar{x}_j} \geq \Delta. \quad (40)$$

Case 2. Both observations are clause observations, corresponding to some clause C_ℓ with $\ell = 1, \ldots, m$.

This means that $s, t \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$. There are 36 possibilities listed below.

How to compute $p_{\chi_1^\ell}q_{\chi_1^\ell}, p_{\chi_2^\ell}q_{\chi_2^\ell}$ and $p_{\chi_3^\ell}q_{\chi_3^\ell}$.
Consider the scalar product $p_{\chi_1^\ell}q_{\chi_1^\ell}$. It is neither affected by goods corresponding to cells with quantity zero nor by goods corresponding to cells in Block 1 and Block 2 with price ε. In Block 1, only goods corresponding to cells in row χ_1^ℓ are considered. The quantity $p_{\chi_1^\ell}q_{\chi_1^\ell}$ contains a part coming from the goods corresponding to cells $(\chi_1^\ell, \chi_1^\ell)$, $(\chi_1^\ell, \chi_2^\ell)$, $(\chi_1^\ell, \chi_3^\ell)$ and (χ_1^ℓ, t_3^ℓ) in Block 1. Each of these goods has a price of one and a quantity of one except the good corresponding to cell $(\chi_1^\ell, \chi_1^\ell)$ which has a price of two and a quantity of one. Therefore, they contribute $2 + 1 + 1 + 1 = 5$ in $p_{\chi_1^\ell}q_{\chi_1^\ell}$. As for goods corresponding to cells in Block 2, we know that there are two goods corresponding to cells (χ_3^ℓ, t_3^ℓ) and (t_3^ℓ, χ_3^ℓ) with price $\frac{1}{2}$ and quantity three. These two goods contribute $2(\frac{1}{2} \times 3) = 3$. In total,

$$p_{\chi_1^\ell}q_{\chi_1^\ell} \cong 5 + 3 = 8. \quad (41)$$

A similar analysis leads to

$$p_{\chi_2^\ell}q_{\chi_2^\ell} \cong 8 \quad (42)$$

and

$$p_{\chi_3^\ell}q_{\chi_3^\ell} \cong 8. \quad (43)$$

How to compute $p_{\chi_1^\ell}q_{\chi_2^\ell}, p_{\chi_2^\ell}q_{\chi_3^\ell}$ and $p_{\chi_3^\ell}q_{\chi_1^\ell}$.
The scalar product $p_{\chi_1^\ell}q_{\chi_2^\ell}$ is affected only by the good of Block 1 corresponding to cell $(\chi_1^\ell, \chi_2^\ell)$. In fact, this good is in row χ_1^ℓ, and therefore gets the price of one in observation χ_1^ℓ. Moreover, the observation χ_2^ℓ uses six units of that good. It is not difficult to see that the goods corresponding to the remaining cells in row χ_1^ℓ of Block 1 get the quantity zero for observation χ_2^ℓ and the goods (χ_3^ℓ, t_3^ℓ) and (t_3^ℓ, χ_3^ℓ) which are the only goods of Block 2 with non-ε price for observation χ_1^ℓ have a quantity of zero for observation χ_2^ℓ; therefore do not contribute in $p_{\chi_1^\ell}q_{\chi_2^\ell}$. Thus

$$p_{\chi_1^\ell}q_{\chi_2^\ell} \cong 6. \quad (44)$$

Similarly, we get

$$p_{\chi_2^\ell}q_{\chi_3^\ell} \cong 6 \quad (45)$$

and

$$p_{\chi_3^\ell}q_{\chi_1^\ell} \cong 6. \quad (46)$$

How to compute $p_{\chi_1^\ell}q_{\chi_3^\ell}, p_{\chi_2^\ell}q_{\chi_1^\ell}$ and $p_{\chi_3^\ell}q_{\chi_2^\ell}$.
The scalar product $p_{\chi_1^\ell}q_{\chi_3^\ell}$ is affected only by the good of Block 1 corresponding to cell $(\chi_1^\ell, \chi_3^\ell)$. That good is in row χ_1^ℓ therefore gets the price of one for observation χ_1^ℓ. Moreover, the observation χ_3^ℓ uses four units of that good. Thus

$$p_{\chi_1^\ell}q_{\chi_3^\ell} \cong 4. \quad (47)$$

Similarly, we get

$$p_{\chi_2^\ell}q_{\chi_1^\ell} \cong 4 \quad (48)$$

and

$$p_{\chi_3^\ell}q_{\chi_2^\ell} \cong 4. \quad (49)$$

How to compute $p_{t_1^\ell}q_{t_1^\ell}, p_{t_2^\ell}q_{t_2^\ell}$ and $p_{t_3^\ell}q_{t_3^\ell}$.
The observation t_1^ℓ is associated to χ_1^ℓ. There are two options for the literal χ_1^ℓ, either $\chi_1^\ell = x_i$ or $\chi_1^\ell = \bar{x}_i$.
If $\chi_1^\ell = x_i$ then the quantity $p_{t_1^\ell}q_{t_1^\ell}$ contains a part coming from the goods corresponding to cells (t_1^ℓ, t_1^ℓ), (t_1^ℓ, χ_2^ℓ) and $(t_1^\ell, \bar{x}_i)$ in Block 1. The first good has a price of two and a quantity of one, while the two others have a price of one and a quantity of one. Therefore, they account for $2 + 1 + 1 = 4$ in $p_{t_1^\ell}q_{t_1^\ell}$. In Block 2, the two goods corresponding to cells $(\chi_2^\ell, \bar{x}_i)$ and $(\bar{x}_i, \chi_2^\ell)$ with price $\frac{1}{2}$ and quantity two are the only goods contributing to

$p_{t_1^\ell} q_{t_1^\ell}$. They contribute for $2(\frac{1}{2} \times 2) = 2$. In sum, $p_{t_1^\ell} q_{t_1^\ell} \cong 4 + 2 = 6$.

On the other hand, if $\chi_1^\ell = \bar{x}_i$ then the quantity $p_{t_1^\ell} q_{t_1^\ell}$ contains a part coming from the goods corresponding to cells (t_1^ℓ, t_1^ℓ), (t_1^ℓ, χ_2^ℓ) and (t_1^ℓ, x_i) in Block 1. As above, these goods account for $2 + 1 + 1 = 4$ in $p_{t_1^\ell} q_{t_1^\ell}$. As for goods in Block 2, only two goods corresponding to cells (χ_2^ℓ, x_i) and (x_i, χ_2^ℓ) with price $\frac{1}{2}$ and quantity two contribute to $p_{t_1^\ell} q_{t_1^\ell}$. They contribute for $2(\frac{1}{2} \times 2) = 2$. In total, we obtain $p_{t_1^\ell} q_{t_1^\ell} \cong 4 + 2 = 6$.

To summarize, whether $\chi_1^\ell = x_i$ or $\chi_1^\ell = \bar{x}_i$, we have

$$p_{t_1^\ell} q_{t_1^\ell} \cong 6. \tag{50}$$

We also obtain, using similar reasoning that

$$p_{t_2^\ell} q_{t_2^\ell} \cong 6 \tag{51}$$

and

$$p_{t_3^\ell} q_{t_3^\ell} \cong 6. \tag{52}$$

How to compute $p_s q_t$ when $s, t \in \{t_1^\ell, t_2^\ell, t_3^\ell\}$ with $s \neq t$.

The scalar product $p_s q_t$ is greater than or equal to the contribution of the good corresponding to cell (s, t) in Block 1. However, the cell (s, t) being in row s of Block 1, it has the price of one for observation s. But that cell is in column t in Block 1 and gets the value Δ as quantity. Therefore

$$p_s q_t \geq \Delta. \tag{53}$$

The set of inequalities (53) represents six values of $p_s q_t$.

How to compute $p_{\chi_1^\ell} q_{t_3^\ell}$, $p_{\chi_2^\ell} q_{t_1^\ell}$ and $p_{\chi_3^\ell} q_{t_2^\ell}$.

The scalar product $p_{\chi_1^\ell} q_{t_3^\ell}$ is determined only by the good corresponding to cell (χ_1^ℓ, t_3^ℓ) in Block 1. This good has a price of one for observation χ_1^ℓ and a quantity of three for observation t_3^ℓ. Therefore, it accounts for $1 \times 3 = 3$ in $p_{\chi_1^\ell} q_{t_3^\ell}$. Notice that the good corresponding to cell (χ_1^ℓ, t_3^ℓ) is the only good of observation t_3^ℓ in row χ_1^ℓ of Block 1 with non-zero quantity. As for goods corresponding to cells in Block 2, we know that the two goods of observation t_3^ℓ in Block 2 with non-zero quantity have the price of ε for observation χ_1^ℓ. Therefore,

$$p_{\chi_1^\ell} q_{t_3^\ell} \cong 3. \tag{54}$$

The following similar results hold

$$p_{\chi_2^\ell} q_{t_1^\ell} \cong 3, \tag{55}$$

$$p_{\chi_3^\ell} q_{t_2^\ell} \cong 3. \tag{56}$$

How to compute $p_{t_3^\ell} q_{\chi_1^\ell}$, $p_{t_1^\ell} q_{\chi_2^\ell}$ and $p_{t_2^\ell} q_{\chi_3^\ell}$.

The scalar product $p_{t_3^\ell} q_{\chi_1^\ell}$ is mainly determined by the good corresponding to cell (t_3^ℓ, χ_1^ℓ) in Block 1. This good has a price of one for observation t_3^ℓ and a quantity of two for observation χ_1^ℓ. Therefore, it accounts for $1 \times 2 = 2$ in $p_{t_3^\ell} q_{\chi_1^\ell}$ and

$$p_{t_3^\ell} q_{\chi_1^\ell} \cong 2. \tag{57}$$

The following similar results hold.

$$p_{t_1^\ell} q_{\chi_2^\ell} \cong 2, \tag{58}$$

$$p_{t_2^\ell} q_{\chi_3^\ell} \cong 2. \tag{59}$$

How to compute $p_s q_t$ and $p_t q_s$ when $s \in \{t_1^\ell, t_2^\ell, t_3^\ell\}$, $t \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$ and $(s, t) \notin \{(t_1^\ell, \chi_2^\ell), (t_2^\ell, \chi_3^\ell), (t_3^\ell, \chi_1^\ell)\}$, $(t, s) \notin \{(\chi_2^\ell, t_1^\ell), (\chi_3^\ell, t_2^\ell), (\chi_1^\ell, t_3^\ell)\}$.

The scalar product $p_s q_t$ is at least as large as the contribution of the good corresponding to cell (s, t) in Block 1. However, the cell (s, t) being in row s of Block 1, it has the price of one for observation s. But that cell is in column t of Block 1 and gets the value Δ as quantity for observation t. Therefore

$$p_s q_t \geq \Delta. \tag{60}$$

Similarly, we prove that

$$p_t q_s \geq \Delta. \tag{61}$$

These are 12 additional scalar products; completing the description of the 36 scalar products announced.

Case 3. One observation is a variable observation, the other is a clause observation such that the corresponding clause does not contain that variable.

This means one observation is in $\{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$ from clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell)$ while the other is a variable observation x_i or $\bar{x}_i$ $(i = 1, 2, \ldots, n)$ which is such that x_i or $\bar{x}_i$ is not in $\{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$. Let $s \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$ and t be a variable observation satisfying the above condition.

How to compute $p_t q_s$ and $p_s q_t$.

The value of the scalar product $p_t q_s$ is least as large as the contribution of good corresponding to cell (t, s) in Block 1. Since the price of that good equals 1, and its quantity equals Δ, we get

$$p_t q_s \geq \Delta. \tag{62}$$

Using similar arguments, we can prove that

$$p_s q_t \geq \Delta. \tag{63}$$

Case 4. One observation is a variable observation, the other is a clause observation such that the corresponding clause contains that variable.

This means one observation is in $\{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$ from clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell)$ while the other is a variable observation x_i or $\bar{x}_i$ which is such that x_i or $\bar{x}_i$ is in $\{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$. Let $s \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell, t_3^\ell\}$ and t be a variable observation satisfying the above condition.

How to compute $p_{x_i} q_{t_1^\ell}$ and $p_{t_1^\ell} q_{x_i}$, when $\chi_1^\ell = x_i$.

The value of the scalar product $p_{x_i} q_{t_1^\ell}$ is at least as large as the contribution of good corresponding to cell (x_i, t_1^ℓ) in Block 1. Since the price of that good equals 1, and its quantity equals Δ, we get

$$p_{x_i} q_{t_1^\ell} \geq \Delta. \tag{64}$$

Using similar arguments, we get

$$p_{t_1^\ell} q_{x_i} \geq \Delta. \tag{65}$$

Similar inequalities hold when $\chi_2^\ell = x_j$ and $\chi_3^\ell = x_k$. These are

$$p_{x_j} q_{t_2^\ell} \geq \Delta, \tag{66}$$

$$p_{t_2^\ell} q_{x_j} \geq \Delta, \tag{67}$$

$$p_{x_k} q_{t_3^\ell} \geq \Delta, \tag{68}$$

$$p_{t_3^\ell} q_{x_k} \geq \Delta. \tag{69}$$

How to compute $p_{\bar{x}_i} q_{t_1^\ell}$ and $p_{t_1^\ell} q_{\bar{x}_i}$ when $\chi_1^\ell = x_i$.

The scalar product $p_{\bar{x}_i} q_{t_1^\ell}$ is not affected by goods corresponding to cells in Block 2. The scalar product $p_{\bar{x}_i} q_{t_1^\ell}$ is determined by the good corresponding to cell $(\bar{x}_i, t_1^\ell)$ in Block 1. That good has a price of one and a quantity of $|\Gamma_{x_i}| + 1$. Therefore, it accounts for $1 \times (|\Gamma_{x_i}| + 1) = |\Gamma_{x_i}| + 1$ in $p_{\bar{x}_i} q_{t_1^\ell}$. Hence,

$$p_{\bar{x}_i} q_{t_1^\ell} \cong |\Gamma_{x_i}| + 1. \tag{70}$$

The scalar product $p_{t_1^\ell} q_{\bar{x}_i}$ is determined by the contribution of good corresponding to cell $(t_1^\ell, \bar{x}_i)$ in Block 1. That good has a

Table 7
Summary of scalar products with values less than Δ.

Id.	Product		Value		Proof
(A)	$p_{x_i}q_{x_i}$	=	$\begin{cases}3 & \text{if } \Gamma_{\bar{x}_i}=\emptyset\\ 2\lvert\Gamma_{\bar{x}_i}\rvert+4 & \text{if } \Gamma_{\bar{x}_i}\neq\emptyset\end{cases}$	$i=1,\ldots,n$	(33)
(B)	$p_{\bar{x}_i}q_{\bar{x}_i}$	=	$\begin{cases}3 & \text{if } \Gamma_{x_i}=\emptyset\\ 2\lvert\Gamma_{x_i}\rvert+4 & \text{if } \Gamma_{x_i}\neq\emptyset\end{cases}$	$i=1,\ldots,n$	(34)
(C)	$p_{x_i}q_{\bar{x}_i}$	=	$\lvert\Gamma_{\bar{x}_i}\rvert+1$	$i=1,\ldots,n$	(35)
(D)	$p_{\bar{x}_i}q_{x_i}$	=	$\lvert\Gamma_{x_i}\rvert+1$	$i=1,\ldots,n$	(36)
(E)	$p_{\chi_r^\ell}q_{\chi_r^\ell}$	=	8	$\ell=1,\ldots,m, r=1,2,3$	(41)–(43)
(F)	$p_{t_r^\ell}q_{t_r^\ell}$	=	6	$\ell=1,\ldots,m, r=1,2,3$	(50)–(52)
(G)	$p_{\chi_i^\ell}q_{\chi_j^\ell}$	=	6	$\ell=1,\ldots,m, (i,j)\in\{(1,2),(2,3),(3,1)\}$	(44)–(46)
(H)	$p_{\chi_i^\ell}q_{\chi_j^\ell}$	=	4	$\ell=1,\ldots,m, (i,j)\in\{(1,3),(2,1),(3,2)\}$	(47)–(49)
(I)	$p_{\chi_i^\ell}q_{t_j^\ell}$	=	3	$\ell=1,\ldots,m, (i,j)\in\{(1,3),(2,1),(3,2)\}$	(54)–(56)
(J)	$p_{t_i^\ell}q_{\chi_j^\ell}$	=	2	$\ell=1,\ldots,m, (i,j)\in\{(3,1),(1,2),(2,3)\}$	(57)–(59)
(K)	$p_{\bar{x}_i}q_{t_r^\ell}$	=	$\lvert\Gamma_{x_i}\rvert+1$ if $\chi_r^\ell=x_i$	$\ell=1,\ldots,m, r=1,2,3, i=1,\ldots,n$	(70), (72) and (74)
(L)	$p_{t_r^\ell}q_{\bar{x}_i}$	=	2 if $\chi_r^\ell=x_i$	$\ell=1,\ldots,m, r=1,2,3, i=1,\ldots,n$	(71), (73) and (75)
(M)	$p_{x_i}q_{t_r^\ell}$	=	$\lvert\Gamma_{\bar{x}_i}\rvert+1$ if $\chi_r^\ell=\bar{x}_i$	$\ell=1,\ldots,m, r=1,2,3, i=1,\ldots,n$	(82), (84) and (86)
(N)	$p_{t_r^\ell}q_{x_i}$	=	2 if $\chi_r^\ell=\bar{x}_i$	$\ell=1,\ldots,m, r=1,2,3, i=1,\ldots,n$	(83), (85) and (87)

price of one and a quantity of two. Thus,

$$p_{t_1^\ell}q_{\bar{x}_i} \cong 2. \tag{71}$$

The inequalities similar to those above hold for the pair of observations t_2^ℓ and $\bar{x}_j$; and t_3^ℓ and $\bar{x}_k$ when $\chi_2^\ell = x_j$ and $\chi_3^\ell = x_k$. There are given by

$$p_{\bar{x}_j}q_{t_2^\ell} \cong |\Gamma_{x_j}|+1, \tag{72}$$

$$p_{t_2^\ell}q_{\bar{x}_j} \cong 2, \tag{73}$$

$$p_{\bar{x}_k}q_{t_3^\ell} \cong |\Gamma_{x_k}|+1, \tag{74}$$

$$p_{t_3^\ell}q_{\bar{x}_k} \cong 2. \tag{75}$$

How to compute $p_{\bar{x}_i}q_{t_1^\ell}$ and $p_{t_1^\ell}q_{\bar{x}_i}$ when $\chi_1^\ell = \bar{x}_i$.
Inequalities similar to those obtained when $\chi_1^\ell = x_i$ hold. These are

$$p_{\bar{x}_i}q_{t_1^\ell} \geq \Delta, \tag{76}$$

$$p_{t_1^\ell}q_{\bar{x}_i} \geq \Delta. \tag{77}$$

If $\chi_2^\ell = \bar{x}_j$ and $\chi_3^\ell = \bar{x}_k$ then

$$p_{\bar{x}_j}q_{t_2^\ell} \geq \Delta, \tag{78}$$

$$p_{t_2^\ell}q_{\bar{x}_j} \geq \Delta, \tag{79}$$

$$p_{\bar{x}_k}q_{t_3^\ell} \geq \Delta, \tag{80}$$

$$p_{t_3^\ell}q_{\bar{x}_k} \geq \Delta. \tag{81}$$

How to compute $p_{x_i}q_{t_1^\ell}$ and $p_{t_1^\ell}q_{x_i}$ when $\chi_1^\ell = \bar{x}_i$.

$$p_{x_i}q_{t_1^\ell} \cong |\Gamma_{\bar{x}_i}|+1, \tag{82}$$

$$p_{t_1^\ell}q_{x_i} \cong 2, \tag{83}$$

$$p_{x_j}q_{t_2^\ell} \cong |\Gamma_{\bar{x}_j}|+1, \tag{84}$$

$$p_{t_2^\ell}q_{x_j} \cong 2, \tag{85}$$

$$p_{x_k}q_{t_3^\ell} \cong |\Gamma_{\bar{x}_k}|+1, \tag{86}$$

$$p_{t_3^\ell}q_{x_k} \cong 2. \tag{87}$$

How to compute $p_{x_i}q_{t_r^\ell}$ and $p_{\bar{x}_i}q_{t_r^\ell}$ when $r \in \{1,2,3\}$ and $\chi_r^\ell \notin \{x_i, \bar{x}_i\}$.
The value of the scalar product $p_{x_i}q_{t_r^\ell}$ is least as large as the contribution of good corresponding to cell (x_i, t_r^ℓ) in Block 1. Since the price of that good equals 1, and its quantity equals Δ, we get

$$p_{x_i}q_{t_r^\ell} \geq \Delta. \tag{88}$$

Similarly,

$$p_{t_r^\ell}q_{x_i} \geq \Delta, \tag{89}$$

$$p_{\bar{x}_i}q_{t_r^\ell} \geq \Delta, \tag{90}$$

and

$$p_{t_r^\ell}q_{\bar{x}_i} \geq \Delta. \tag{91}$$

How to compute p_sq_t and p_tq_s when $s \in \{\chi_1^\ell, \chi_2^\ell, \chi_3^\ell\}$ and t is a variable observation.
It is not difficult to prove that all these scalar products are at least as large as Δ. That is

$$p_sq_t \geq \Delta \tag{92}$$

and

$$p_tq_s \geq \Delta. \tag{93}$$

Case 5. Both observations are clause observations; one of them from clause C_{ℓ_1}, the other from clause C_{ℓ_2} with $\ell_1 \neq \ell_2$.

Let $s \in \{\chi_1^{\ell_1}, \chi_2^{\ell_1}, \chi_3^{\ell_1}, t_1^{\ell_1}, t_2^{\ell_1}, t_3^{\ell_1}\}$ and $t \in \{\chi_1^{\ell_2}, \chi_2^{\ell_2}, \chi_3^{\ell_2}, t_1^{\ell_2}, t_2^{\ell_2}, t_3^{\ell_2}\}$. These are 36 pairs of observations (s,t).

How to compute p_sq_t and p_tq_s.
It is not difficult to obtain the following lower bounds.

$$p_sq_t \geq \Delta \tag{94}$$

and

$$p_tq_s \geq \Delta. \tag{95}$$

Notice that for two distinct observations s and t in $\mathbb{T}$, we have $p_sq_t \geq \Delta$ if and only if $p_tq_s \geq \Delta$.

In Table 7, we summarize the scalar products computed above by presenting only those which have values less than Δ.

Appendix B. Proof of Claim 1

In this section, we prove Claim 1.

We are now in a position to finish the proof of Claim 1. Here we show how the inequalities identified by Claim 1 follow from the scalar product computed in Appendix A.

The first set of inequalities (1) comes from (A) and (C) in Table 7.
The inequalities (2) stem from (B) and (D) in Table 7.
The inequalities (3), (7) and (9) stem from (E) and (G) in Table 7.
The inequalities (4), (6) and (10) stem from (E) and (H) in Table 7.

The inequalities (5), (8) and (11) stem from (E) and (I) in Table 7.

The inequalities (13), (16) and (19) stem from either (F) and (L) or (F) and (N), in Table 7.

The inequalities (12), (15) and (18) stem from (F) and (J) in Table 7.

The inequalities (14), (17) and (20) stem from either (B) and (K) or (A) and (M), in Table 7.

The set of inequalities (21) come from the fact that for any other pair of observations $s, t \in \mathbb{T}$, the scalar product $p_s q_t$ is greater than or equal to Δ.

Appendix C. Proof of Claim 2

In this section, we prove Claim 2.

Here, we show how the double-sum inequalities described by Claim 2 originate from the scalar products computed in Appendix A. For every clause $C_\ell = (\chi_1^\ell \vee \chi_2^\ell \vee \chi_3^\ell) \in C$, $\ell \in \{1, \dots, m\}$ with the given clause observations $\chi_1^\ell, \chi_2^\ell, \chi_3^\ell, t_1^\ell, t_2^\ell$ and t_3^ℓ, we have:

The double-sum inequalities (22)–(24) come from (E), (H) and (I) in Table 7.

The inequalities (25), (27) and (29) stem from (F), (J) and either (L) or (N) in Table 7.

The inequalities (26), (28) and (30) stem from either (B), (D) and (K) or (A), (C) and (M) in Table 7.

The inequalities (31) stem from (A) and (M) and the inequalities (32) from (B) and (K) in Table 7.

The non-existence of the other possible inequalities is justified by the fact that for those inequalities, at least one scalar product appearing in the right-hand side has a value greater than or equal to Δ.

References

Cherchye, L., De Rock, B., Vermeulen, F., 2007. The collective model of household consumption: a nonparametric characterization. Econometrica 75, 553–574.

Cherchye, L., De Rock, B., Sabbe, J., Vermeulen, F., 2008. Nonparametric tests of collectively rational consumption behavior: an integer programming procedure. Journal of Econometrics 147, 258–265.

Garey, M.R., Johnson, D.S., 1979. Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman and Co.

Talla Nobibon, F., Cherchye, L., De Rock, B., Sabbe, J., Spieksma, F.C.R., 2010. Heuristics for deciding collectively rational consumption behavior. Computational Economics, (doi:10.1007/s10614-010-9228-9).

Varian, H., 1982. The nonparametric approach to demand analysis. Econometrica 50, 945–974.