Mathematics of Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

How to Design a Stable Serial Knockout Competition

Roel Lambers, Rudi Pendavingh, Frits Spieksma

Published online in Articles in Advance 10 Jun 2024

. https://doi.org/10.1287/moor.2022.0352

informs
https://pubsonline.informs.org/journal/moor
MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–12
ISSN 0364-765X (print), ISSN 1526-5471 (online)

How to Design a Stable Serial Knockout Competition

Roel Lambers,[a,*] Rudi Pendavingh,[b] Frits Spieksma[b]

[a] Faculteit Business en Economie, Amsterdam University of Applied Sciences, 1102 CV Amsterdam, Netherlands; [b] Mathematics & Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, Netherlands

*Corresponding author

Contact: r.lambers2@hva.nl, https://orcid.org/0000-0002-0314-6094 (RL); r.pendavingh@tue.nl (RP); f.c.r.spieksma@tue.nl (FS)

Received: December 20, 2022
Revised: May 30, 2023
Accepted: October 8, 2023
Published Online in Articles in Advance: June 10, 2024

MSC2020 Subject Classifications: Primary: 13P25; Secondary: 05-11

https://doi.org/10.1287/moor.2022.0352

Abstract. We investigate a new tournament format that consists of a series of individual knockout tournaments; we call this new format a serial knockout competition (SKC). This format has recently been adopted by the Professional Darts Corporation. Depending on the seedings of the players used for each of the knockout tournaments, players can meet in the various rounds (e.g., first round, second round … semifinal, final) of the knockout tournaments. Following a fairness principle of treating all players equal, we identify an attractive property of an SKCl each pair of players should potentially meet equally often in each of the rounds of the SKC. If the seedings are such that this property is indeed present, we call the resulting SKC *stable*. In this note, we formalize this notion, and we address the following question. Do there exist seedings for each of the knockout tournaments such that the resulting SKC is stable? We show using a connection to the Fano plane that the answer is yes for eight players, and we prove that the resulting SKC is unique up to permutations of the players. We further prove that stable SKCs exist for any numbers of players that are a power of two, and we provide stable schedules for competitions on 16 and 32 players.

Funding: The research of F. Spieksma was partly funded by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek [Grant 024.002.003].

Keywords: darts • combinatorics • Galois field • optimization • fairness

1. Introduction

Two popular tournament formats are the round-robin format and the knockout format. In a round-robin format, each pair of players (or teams) meets a given number of times. In a knockout tournament, starting from a so-called *seeding*, each round of the knockout tournament sees matches between all remaining players, and a player is removed from the tournament after losing a match; in this way, after $\log n$ rounds, a winner is determined (where n is the number of players).

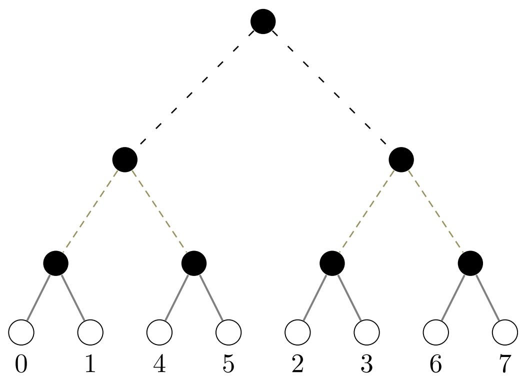
Each of these formats has been studied intensely from very different viewpoints. In particular, deciding upon a seeding of the players in a single-knockout tournament has attracted a lot of attention; we do not aim to review this field, and we simply refer to Aziz et al. [1], Groh et al. [6], Horen and Riezman [7], Karpov [8], Manurangsi and Suksompong [9], Vu [10], Vu and Shoham [11], and the references contained therein for more information on this subject. Most of this literature assumes that probabilities are given that denote the chance of one player beating the other.

In practice, it is not uncommon to design a tournament combining both formats; for instance, first, have a number of round-robin tournaments in parallel, and then, let the winners of the round-robins participate in a knockout tournament.

In this note, we study a new format that can be seen as an alternative combination of a knockout tournament and a round-robin tournament. Let the number of players n be equal to 2^k for some $k \geq 2$, allowing us to focus exclusively on so-called *balanced* knockout tournaments (i.e., knockout tournaments where each player has to play the same number of matches to win the tournament). Observe that a balanced knockout tournament consists of k successive *rounds*, where in round I, the remaining 2^{k+1-i} players compete, $i = 1, \ldots, k$.

The competition format we study consists of a set of $2^k - 1$ knockout tournaments. We will call this format a *serial knockout competition* (SKC for short). Related (but different) formats are the so-called quasidouble-knockout tournament (Considine and Gallagher [3]) and the multiple-elimination knockout tournament (Fayers [5]). The problem that we analyze in this note is to specify for each of the individual knockout tournaments that make up the SKC the *seeding*; these seedings specify, for each player, the leaf nodes of the underlying knockout trees to which the player is assigned. See Figure 1 for an example of a single-knockout tournament.

Figure 1. A single knockout T, where players $0, 1, \ldots, 7$ are assigned to the leaf nodes, leading to the seeding $s = 0145\text{-}2367$.



Once the seedings are specified, the individual knockout tournaments of the SKC can unfold; no other decisions in the design of the competition need to be taken. We refer to specifying the seedings as the *design* of the SKC.

In this note, we do not deal with determining the winner of an SKC; instead, we focus on the following question. How do we design an SKC in a fair way?

Here, we interpret fair by asking for a design that (i) treats all players equal without any prior assumptions on the strengths of the players and that (ii) has each pair of players meeting equally often in each of the rounds of an SKC.

One could argue that simply picking random seedings leads to a fair SKC as each player, in expectation, meets each other player equally often. However, it is clear that because of the inherent variability of picking random seedings, a design is found that violates these conditions.

Thus, we aim to find seedings such that over the SKC, each pair of players meets equally often in all rounds. Consider for instance the first round; as the SKC consists of $2^k - 1$ knockout tournaments, each player plays $2^k - 1$ first-round matches. Hence, we want to find seedings such that each player meets each other player exactly once in a first round. More generally, the question is as follows. Do there exist seedings such that each pair of players meets equally often in each of the rounds of the SKC?

We capture this notion formally by defining the notion of *stability* of an SKC.

Definition 1. Given a knockout tournament T for $n = 2^k$ players, we say that $v_T(x, x') = i$ if players x, x' can meet in round i of that tournament, $i = 1, \ldots, k$.

The phrase "can meet" in the definition refers to the assumption that players x and x' win their matches in the rounds prior to their encounter. For instance, in Figure 1, players 1 and 4 can meet in round 2, whereas players 0 and 3 can meet in round 3, which is the final round.

Let us now formally define the concept of stability, where we use $\#S$ to denote the number of elements of a finite set S.

Definition 2. Given a set of knockout tournaments $\mathcal{T}$ on $n = 2^k$ players, we say that it is stable in round i if there is a number c_i so that

$$\#\{T \in \mathcal{T} : v_T(x, x') = i\} = c_i$$

for all pairs of distinct players x, x'. We say that the set $\mathcal{T}$ is stable if it is stable in all rounds $i = 1, \ldots, k$.

Observe that the expression $\#\{T \in \mathcal{T} : v_T(x, x') = i\}$ counts the tournaments T from the set $\mathcal{T}$ such that players x and x' can meet in round i in T, $1 \leq i \leq k$.

Definition 3. We define a serial knockout competition as a competition for $n = 2^k$ players consisting of $n - 1$ knockout tournaments.

Notice that in an individual knockout tournament T, a player can meet any of 2^{i-1} other players when reaching round i (i.e., for each player x, we have $\#\{x' : v_T(x, x') = i\} = 2^{i-1}$, $i = 1, \ldots, k$). As an SKC consists of $2^k - 1$ knockout tournaments, the number of meetings that are possible in round i for any player is given by $(2^k - 1)2^{i-1}$, $1 \leq i \leq k$. With the number of opponents of any player x equal to $n - 1 = 2^k - 1$, an SKC is stable in round i if $c_i = 2^{i-1}$ for $i = 1, \ldots, k$.

In this note, we prove that stable SKCs exist for arbitrary $n = 2^k$. We describe in Section 1.1 the case that motivates this work. In Section 2, we investigate the case of eight players, and in Section 3, we deal with the general case. We illustrate in Section 4 the cases of 16 and 32 players. In Section 5, we prove that a stable SKC is unique for eight players (up to permutations). Section 6 focuses on the relation between the stability of an SKC and the so-called order of play (a property that is relevant for the motivation of this work). We close in Section 7.

1.1. Motivation: The Premier League of Darts

The motivation for investigating this particular tournament design comes from the Professional Darts Corporation (PDC). We now describe this competition in more detail.

The Premier League of Darts, organized by the PDC, is an annual competition where the best darts players of the world compete over several months for the title. The 2022 edition featured the best eight players, and it started on February 3, 2022 and ended on June 13, 2022. Total prize money was £1,000,000, and the winner pocketed £275,000. The concept of the league changed drastically compared with the previous years; this edition consisted of 16 knockout tournaments. Thus, there is a winner for each of these knockout tournaments, and importantly, in every single match, there is something to play for, which adds to the excitement of the format; see Groh et al. [6].

The 16 knockout tournaments are structured in the following way. The first seven knockout tournaments have a predetermined seeding; then, there is a special knockout tournament, again seven knockout tournaments with a given seeding, and a last special knockout tournament. The seedings in the special knockout tournaments depend on the standings at that time. The other (regular) knockout tournaments have a fixed seeding that is determined in advance by the PDC. Our analysis focuses on the seedings in these regular knockout tournaments. The first seven knockout tournaments as well as the second seven regular knockout tournaments each correspond to an SKC.

We point out that neither the first seven knockout tournaments nor knockout tournaments 9–15 of the 2022 edition of the Premier League of Darts are stable SKCs as defined in Definition 2, as can be verified on Wikipedia [12]. Specifically, although it is true that each of these SKCs is stable in round 1 (meaning that each pair of players meets once in the first round in each of these SKCs), there are pairs of players who can never meet in the semifinal.

As far as we are aware, this is the first occurrence of an SKC in practice. One reason explaining why an SKC format is not being used more often in practice is the fact that knockout tournaments are used when a match is physically (or otherwise) demanding and one wants to have relatively few matches to determine a winner. As an SKC requires multiple knockouts, it does not constitute a format with few matches. However, this argument does not apply when the tournament can be organized over a relatively long time period (as in the case of the PDC), and it also does not apply in the domain of e-sports as these require little (physical) effort. E-sports are a fast-growing domain with an enormous amount of competitions being organized. We expect that the format of an SKC, or variations thereof, will turn out to be useful and popular in e-sports as it combines the excitement of a knockout format with the fairness of a round-robin format.

2. Constructing a Stable SKC When $n = 8$

In this section, we are going to construct a stable SKC tournament $\mathcal{T} = (T_r)_{r\leq 7}$ for eight players; this analysis applies directly to the situation encountered by the PDC (see Section 1.1). Each knockout tournament is specified by providing a *seeding* s (i.e., a correspondence between players $0, \ldots, 7$ and places in tournament brackets). In Figure 1, it is shown how to make a knockout tree out of the seeding $s = 01452367$. Although the permutation itself holds all the information needed, we may place hyphens as a visual aid indicating the halves of the seeding: 0145-2367 instead of 01452367 (Figure 2).

Example 1. The permutation 0145-2367 corresponds to the tree in Figure 1.

Figure 2. Knockout tree T with seeding s = 0145-2367.

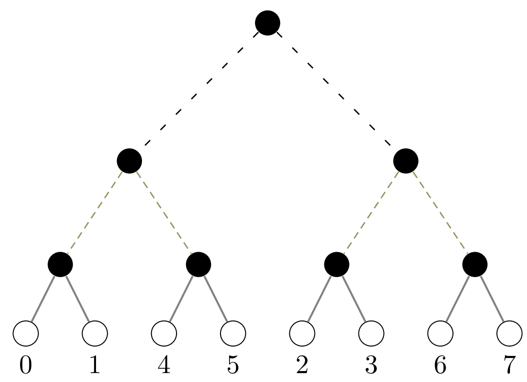


Table 1. Seedings for a stable SKC.

Knockout tournament	Seeding	Node	Line
1	0145-2367	1	145
2	0213-4657	2	123
3	0356-1247	3	356
4	0426-1537	4	246
5	0527-1436	5	275
6	0617-2435	6	176
7	0734-1625	7	347

As for the construction, we first simply state a stable SKC in Table 1, after which we give a method to generate such a set of seedings.

In Table 1, the last two columns refer to nodes and lines. These nodes and lines are elements of the Fano plane used to get to these seedings. This plane is depicted in Figure 3, where players 1–7 are placed on the seven nodes. A line in the Fano-plane is refered to by naming the 3 nodes on the line; line 145 is the line going through nodes 1, 4 and 5. We construct a seeding in the following way.

- Select a node $x \in \{1, \ldots, 7\}$. This indicates that player 0 meets player x in the first knockout tournament. In case $x = 1$, we have a partial seeding $s = 01\ldots$.
- Select a line that goes through node x. The players corresponding to the two other nodes on the line meet each other. In case $x = 1$, if we select the line 145, then player 4 and player 5 meet, and we extend the partial seeding to $s = 0145\ldots$.
- The remaining two matches are given by the two nonselected lines through node x. The two players on each line, respectively, meet each other. This means that in case $x = 1$, players 2 and 3 (line 123) and players 6 and 7 (line 176) meet in the first knockout tournament. The resulting seeding for the first knockout tournament is thus given by 0145-2376.

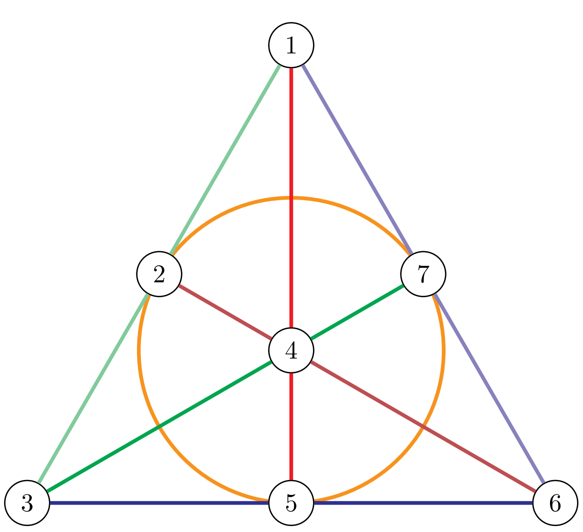
A routine verification shows that the knockout tournament arising from a node and a line has the following key property.

Lemma 1. *Let T be the knockout tournament that arises from the node-line pair x, ℓ of the Fano plane, and let y be a node of the Fano plane. Then,*
- $v_T(0, y) = 1$ *if and only if* $y = x$,
- $v_T(0, y) = 2$ *if and only if* $y \in \ell$ *and* $y \neq x$, *and*
- $v_T(0, y) = 3$ *if and only if* $y \notin \ell$.

Moreover, if $\ell' = \{y, x, x'\}$ is any line of the Fano plane containing the node y, then $v_T(x, x') = v_T(0, y)$.

Notice that in Table 1, each node and each line of the Fano plane occur exactly once, and each node is on the corresponding line. The following theorem states that this construction is sufficient to obtain a stable SKC.

Figure 3. (Color online) The Fano plane used to construct Table 1.



Theorem 1. *Let $x_1, \dots, x_7$ be an enumeration of the nodes and $\ell_1, \dots, \ell_7$ be an enumeration of the lines of the Fano plane such that $x_r \in \ell_r$ for $r = 1, \dots, 7$. Let T_r be the knockout tournament that arises from the pair x_r, ℓ_r. Then, the SKC defined by $\mathcal{T} := \{T_1, \dots, T_7\}$ is stable.*

Proof. To show that $\mathcal{T}$ is stable, we need to show that

$$\#\{T \in \mathcal{T} : v_T(x, x') = i\} = 2^{i-1} \tag{1}$$

for each pair of distinct players x, x' and each round $i \in \{1, 2, 3\}$. Notice that $\mathcal{T}$ is stable in round $i = 3$ if it is stable in both round 1 and round 2.

We first consider the case that one of x, x' is zero, say $\{x, x'\} = \{0, y\}$ for some $y \in \{1, \dots, 7\}$.

- When $i = 1$, our construction ensures that in each individual knockout tournament $r = r^y$, there exists a unique player $x_r = y$ meeting player 0. Hence, $\#\{T \in \mathcal{T} : v_T(0, y) = 1\} = \#\{r : y = x_r\} = 1$, and Equation (1) is satisfied for $i = 1$.
- When $i = 2$, we observe that there are exactly three lines through y; thus, there exist two distinct knockout tournaments $r, r' \neq r^y$ such that $y \in \ell_r, \ell_r'$—meaning that $(0, y)$ can meet in round 2 in those knockout tournaments. Thus, $\#\{T \in \mathcal{T} : v_T(0, y) = 2\} = \#\{r : y \in \ell_r, y \neq x_r\} = 2$, and Equation (1) is satisfied for $i = 2$.

This settles the case where one player is player 0. Next, suppose x, x' are distinct players, and both are not player 0. Then, the Fano plane contains a unique node y and line $\ell' = \{y, x, x'\}$ through x, x'. By Lemma 1, we have $v_T(x, x') = v_T(0, y)$ for each $T \in \mathcal{T}$. As $\#\{T \in \mathcal{T} : v_T(0, y) = i\} = 2^{i-1}$ for all y, this holds for any distinct pair x, x', for $i = 1, 2, 3$. The theorem follows. □

We point out that from the viewpoint of stability, the sequence with which the individual knockout tournaments are played is irrelevant.

3. Constructing a Stable SKC

Here, we generalize the node-line construction used in Section 2 to find a stable SKC for $n = 2^k$ players. In Section 3.1, we describe the basic idea, and in Section 3.2, we make a connection to Galois fields (GFs). We use this connection in Section 3.3 to prove our main result: Theorem 2.

3.1. The Basic Idea

The key idea that we will carry over to the general setting is that we will construct our knockout tournaments in a restricted way so that for each pair of players x, x', there is a well-defined player y such that

$$v_T(x, x') = v_T(0, y)$$

for all knockout tournaments T of this restricted form. Showing that an SKC $\mathcal{T}$ is stable, where each tournament $T \in \mathcal{T}$ is of this special form, it then reduces to verifying that

$$\#\{T \in \mathcal{T} : v_T(0, y)\} = 2^{i-1}$$

for each player y and each round i, $i = 1, \dots, k$.

To define the representative y of a pair of players x, x' and to create the special tournaments T, we need additional structure on the set of players. For the case $n = 8$, we identified the nonzero players with nodes of the Fano plane and used its geometry to define the tournaments. In what follows, we will identify the $n = 2^k$ players with the 2^k elements of the *Galois field* $GF(2^k)$.

As $GF(2^k)$ is a field, both addition and multiplication are possible operations on its elements. We construct a tournament T such that for $x, x' \in GF(2^k)$, we have

$$v_T(x, x') = v_T(0, y)$$

when $y := x - x'$.

After we have constructed a base model for our knockout tournament, we use the multiplication in $GF(2^k)$ on T to create tournaments $T(z)$ for each nonzero element z of $GF(2^k)$, and we argue that

$$\mathcal{T} := \{T(z) : z \neq 0\}$$

is a stable SKC.

3.2. The Connection to Galois Fields

To exploit the structure of Galois field $GF(2^k)$, we first have to describe $GF(2^k)$. Although we do not go into too much detail, we point out the main properties that we use. For an accessible introduction to finite fields, see Chavez and O'Neill [2].

A *binary polynomial* $q \in \mathbb{Z}_2[X]$ is an expression of the form

$$q = q_k X^k + \ldots q_1 X + q_0,$$

where the coefficients q_i are either zero or one. Such polynomials may be added and multiplied as usual, except that the coefficients of these binary polynomials are added according to the rule $1 + 1 = 0$; this applies only to coefficients of binary polynomials. So, that is,

$$(X + 1) \cdot (X^2 + X + 1) = X^3 + X^2 + X^2 + X + X + 1 = X^3 + 1.$$

The degree of a polynomial $q = \sum_i q_i x^i$ is the highest value of i so that $q_i \neq 0$. The polynomial $q = X^3 + 1$ that is the outcome of the calculation is *reducible* because it has degree 3 and is the product of two polynomials of strictly lower degree ($X + 1$ of degree 1 and $X^2 + X + 1$ of degree 2). For any value of k, irreducible polynomials $q \in \mathbb{Z}_2[X]$ are guaranteed to exist. For example, when $k = 3$, the polynomial $q = X^3 + X^2 + 1$ is irreducible over $\mathbb{Z}_2[X]$. Other irreducible polynomials of small degree are $X^2 + X + 1, X^4 + X + 1, X^5 + X^2 + 1$ for degree $k = 2, 4, 5$, respectively.

Given any polynomial $q \in \mathbb{Z}_2[X]$, we write $\mathbb{Z}_2[X]/(q)$ for the set of polynomials one gets from a polynomial in $\mathbb{Z}[X]$ by filling in a symbolic value α that is assumed to satisfy $q(\alpha) = 0$. If $q = X^2 + X + 1$, then the element $x = \alpha^3 \in \mathbb{Z}_2[X]$ can be rewritten as

$$x = \alpha^3 = \alpha^3 + \alpha \cdot q(\alpha) = \alpha^3 + \alpha \cdot (\alpha^2 + \alpha + 1) = \alpha^2 + \alpha = \alpha^2 + \alpha + q(\alpha) = 1$$

because $q(\alpha) = 0$. Indeed, any element $x \in \mathbb{Z}_2[X]/(q)$ can be rewritten to $x = x_{k-1}\alpha^{k-1} + \cdots + x_1\alpha + x_0$: that is, without using powers α^i with $i \geq k$ in the expression.

If $q \in \mathbb{Z}_2[X]$ is an *irreducible* polynomial of degree k, it is known that $GF(2^k) \cong \mathbb{Z}_2[X]/(q)$ is a *field*; one can add and multiply with its elements, but also, one can divide by any nonzero element. Indeed, consider that in the example with $q = X^2 + X + 1$, we had $\alpha \cdot \alpha^2 = \alpha^3 = 1$. Then, $\alpha^{-1} = \alpha^2$, and dividing by α amounts to multiplying with α^2. The irreducibility of q ensures that for any nonzero $x \in GF(2^k)$, there is a $y \in GF(2^k)$ so that $x \cdot y = 1$. Then, a division by x can be executed as a multiplication by y.

There is more than one irreducible polynomial q of each degree k, but whichever one is used, the outcome is mathematically "the same" field $GF(2^k)$. Having fixed a polynomial q for the construction of the Galois field $GF(2^k)$, there is just one way to write an element $x \in GF(2^k)$ as $x = \sum_{i=0}^{k-1} x_i \alpha^i \in GF(2^k)$, and we may define the *degree* of x as $d(x) = \max\{i : x_i \neq 0\}$.

This degree leads us to the following lemma on the existence of a tournament T with the nice property that $v_T(x, y) = v_T(0, x - y) = 1 + d(x - y)$.

Lemma 2. *There is a knockout tournament T whose players are the elements of $GF(2^k)$ so that $v_T(x, y) = 1 + d(x - y)$ for all $x, y \in GF(2^k)$.*

Proof. We construct tournament T by inductively constructing T_m for incremental values $m = 1, \ldots, k$, where each T_m is a knockout tournament on the set $P_m = \{x \in GF(2^k) : d(x) < m\}$, and all the T_m have the property that $v_{T_m}(x, y) = 1 + d(x - y)$ for $x, y \in P_m$. Then, $T = T_k$ proves the lemma.

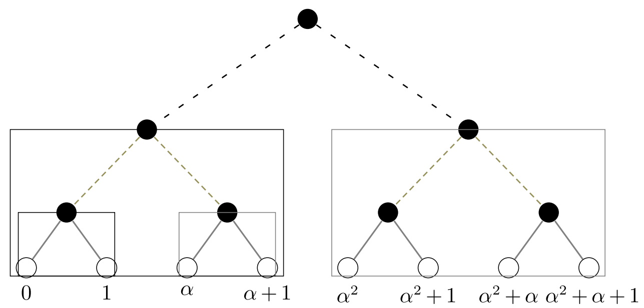
When $m = 1$, the set $P_0 = \{0, 1\}$ contains only two players, and the unique tournament T_1 that one can construct on these two players has $v_{T_1}(0, 1) = 1 = 1 + d(1 - 0)$.

As an induction step, assume that T_m exists such that $v_{T_m}(x, y) = 1 + d(x - y)$ for all $x, y \in P_m$. Let T'_m arise from a copy of T_m by adding α^m to each player. Then, T'_m has players $P'_m = \{x + \alpha^m : x \in P_m\}$, and for any two players $x', y' \in P'_m$, we have

$$v_{T'_m}(x', y') = v_{T_m}(x, y) = 1 + d(x - y) = 1 + d(x' - y'),$$

where $x' = x + \alpha^m$ and $y' = y + \alpha^m$ with $x, y \in P_m$.

Figure 4. A knockout tournament T so that $v_T(x,y) = 1 + d(x-y)$.



We construct T_{m+1} for players $P_{m+1} = P_m \cup P'_m$ as the combination of tournaments T_m, T'_m, where at round $m+1$, the winner of T_m plays the winner of T'_m. For this T_{m+1}, we see that for $x, y \in P_{m+1}$,

$$v_{T_{m+1}}(x,y) = v_{T_m}(x,y) = 1 + d(x-y) \quad \text{if } x,y \in P_m$$
$$v_{T_{m+1}}(x,y) = v_{T'_m}(x,y) = 1 + d(x-y) \quad \text{if } x,y \in P'_m$$
$$v_{T_{m+1}}(x,y) = 1 + m = 1 + d(x-y) \quad \text{if } x \in P_m, y \in P'_m \text{ or } x \in P'_m, y \in P_m.$$

This finishes the induction step. Taking $T = T_k$ gives the desired tournament. □

The construction of T with elements in $GF(2^3)$ is given in Figure 4.

3.3. The Result

By Lemma 2, we know that there exists a knockout tournament T on the elements of $GF(2^k)$ such that $v_T(x,y) = v_T(0, x-y) = 1 + d(x,y)$ for all $x, y \in GF(2^k)$. In the following section, we argue that for each nonzero $z \in GF(2^k)$, the tournament $T(z)$ obtained from T by replacing each player x by zx maintains the property that $v_{T(z)}(x,y) = v_{T(z)}(0, x-y)$. Then, we show that

$$\mathcal{T} = \{T(z) : z \in GF(2^k) \setminus \{0\}\}$$

is a stable SKC.

Let T be a tournament satisfying Lemma 2; thus, $v_T(x,y) = 1 + d(x-y)$ for all $x, y \in GF(2^k)$. Let $z \in GF(2^k)$ be nonzero and thus, invertible. We construct $T(z)$ from T by replacing each player x with zx. As the map $x \longmapsto zx$ is one to one, $T(z)$ is again a tournament whose players are the elements of $GF(2^k)$. Evidently, we have $v_{T(z)}(x,y) = v_T(z^{-1}x, z^{-1}y)$ for all $x, y \in GF(2^k)$. It follows that

$$v_{T(z)}(x,y) = v_T(z^{-1}x, z^{-1}y) = v_T(0, z^{-1}(x-y)) = v_{T(z)}(0, x-y)$$

for all $x, y \in GF(2^k)$ and

$$v_{T(z)}(0,y) = v_T(0, z^{-1}y) = 1 + d(z^{-1}y)$$

for all $y \in GF(2^k)$.

Theorem 2. *$\mathcal{T} := \{T(z) : z$ a nonzero element of $GF(2^k)\}$ is a stable SKC.*

Proof. We need to show that $\#\{T \in \mathcal{T} : v_T(x,x') = i\} = 2^i$ for each pair of distinct players $x, x' \in GF(2^k)$ and each round $i = 1, \ldots, k$.

If one of x, x' is zero, say $\{x, x'\} = \{0, y\}$ with $y \neq 0$, then, for each $i = 1, \ldots, k$,

$$\#\{T \in \mathcal{T} : v_T(0,y) = i\} = \#\{z \in GF(2^k) : z \neq 0, 1 + d(z^{-1}y) = i\}.$$

Substituting z by $r^{-1}y$, this equals

$$\#\{r^{-1}y \in GF(2^k) : r \neq 0, 1 + d(r) = i\} =$$
$$\#\{r \in GF(2^k) : r \neq 0, 1 + d(r) = i\} = 2^i$$

because the map $r \longmapsto r^{-1}y$ is one to one.

Table 2. Multiplication on $GF(2^3)$.

	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	0	0	0	0	0	0	0
1	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
α	α	α^2	$\alpha^2+\alpha$	$\alpha+1$	1	$\alpha^2+\alpha+1$	α^2+1
$\alpha+1$	$\alpha+1$	$\alpha^2+\alpha$	α^2+1	$\alpha^2+\alpha+1$	α^2	1	α
α^2	α^2	$\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α	α^2+1	1
α^2+1	α^2+1	1	α^2	α	$\alpha^2+\alpha+1$	$\alpha+1$	$\alpha^2+\alpha$
$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	1	α^2+1	$\alpha+1$	α	α^2
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	α^2+1	α	1	$\alpha^2+\alpha$	α^2	$\alpha+1$

The general case reduces to this special case because each of the tournaments $T \in \mathcal{T}$ has $v_T(x,x') = v_T(0, x-x')$. Then,

$$\#\{T \in \mathcal{T} : v_T(x,x') = i\} = \#\{T \in \mathcal{T} : v_T(0, x-x') = i\} = 2^i$$

as required. □

We close this section with an example that constructs a stable SKC on eight players using the Galois group.

Example 2. For the Galois group, we choose $q(X) = X^3 + X + 1$ as the irreducible polynomial over $\mathbb{Z}_2$ and set $q(\alpha) = 0$. The corresponding multiplication table is shown in Table 2.

Table 2 essentially gives the seedings for the SKC because the row for multiplication by z presents the seeding for $T(z)$. Upon replacing each polynomial with the number specified in Table 3, we get the SKC of Table 4.

Comparing the SKC from Table 1 with the one shown in Table 4, we see that the knockout tournaments are the same and merely permuted.

4. Stable SKC on 16 and 32 Players

In this section, we use the construction of the previous section to generate an SKC on 16 players and an SKC on 32 players. For notational purposes, we enumerate the first 10 players by $0, \dots, 9$ and continue with a, b up until f in the case of 16 teams and v in the case of 32 teams. By doing this, we can visualize the seedings as a string of length 16 (32), where each character is one player.

The seedings are shown in Tables 5 and 6.

5. Equivalence and Uniqueness of SKCs

5.1. Equivalence of SKCs

Two tournaments T and T' on players P are considered identical if $v_T(p,q) = v_{T'}(p,q)$ for all pair of players $p, q \in P$. For any tournament T on players P and any bijection $s : P \longrightarrow P'$ between sets of players P and P', let the tournament $T(s)$ on players P' arise from T by replacing each player $p \in P$ in T with player $s(p) \in P'$ (note that this notation generalizes the one we used in previous sections, considering that an element $z \in GF(2^k)$ uniquely determines a bijection $x \longmapsto zx$, and it is this bijection that, in turn, determines the tournament $T(z)$). Evidently, if $|P| = |P'|$, then any two tournaments T, T' on P (P') are *equivalent* in the sense that $T(s) = T'$ for some bijection $s : P \longrightarrow P'$. So, fixing some tournament T on P, the set

$$\{T(s) : s \text{ a bijection from } P \text{ to } P'\}$$

is the set of *all* tournaments on P'. If $n := |P| = |P'|$, then there are $n!$ bijections $s : P \longrightarrow P'$, and for each T' on P', there are 2^{n-1} bijections s so that $T' = T(s)$. It follows that there are $n!/2^{n-1}$ different tournaments on any given set of n players.

An SKC is defined as an (unordered) set of tournaments, and so, two SKCs are mathematically identical if they contain the same tournaments. We consider SKCs $\mathcal{T}$ on P and $\mathcal{T}'$ on P' as *equivalent* if one is obtained from the other by consistently replacing each player from P with a player in P': that is, if there is a bijection $s : P \longrightarrow P'$ so that $\mathcal{T}' = \mathcal{T}(s) := \{T(s) : T \in \mathcal{T}\}$.

Table 3. From Galois to teams.

0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	1	4	5	2	3	6	7

Table 4. SKC constructed from Table 2.

Knockout tournament	Seeding
1	0145-2367
2	0257-6431
3	0312-4756
4	0426-5173
5	0563-7214
6	0671-3542
7	0734-1625

Given any SKC $\mathcal{T}$ on players P, the organizer of a tournament for players P' has the freedom to choose a bijection $s : P \to P'$ to obtain an SKC $\mathcal{T}(s)$ on P'. The number of different SKCs that arise from $\mathcal{T}$ in this manner will depend on the internal symmetry of $\mathcal{T}$. The stable SKC $\mathcal{T}$ on players $GF(2^k)$ from the construction in this paper has the property that for any two bijections s, s', we either have $\mathcal{T}(s) = \mathcal{T}(s')$ or have $\mathcal{T}(s) \cap \mathcal{T}(s') = \emptyset$. As a result, the set of SKCs

$$\{\mathcal{T}(s) : s \text{ is a bijection } GF(2^k) \to P\}$$

partitions the set of $n!/2^{n-1}$ different tournaments on n players into parts of size $n-1$. It follows that there are $\frac{n!/2^{n-1}}{n-1}$ different ways to assign players to this SKC. In particular, from the stable SKC on $n=8$ players constructed in this paper, we can obtain 45 different stable SKCs.

5.2. Uniqueness of Stable SKCs up to Equivalence

We have investigated the possibility that any two stable SKCs on $n = 2^k$ players are equivalent. On the positive side, one readily verifies that for $n=2$ and $n=4$ players, there is a unique stable SKC, making the equivalence of stable SKCs trivially true in these cases. We can also manage the case of $n=8$. Let $\mathcal{T}_F$ denote the stable SKC obtained from the Fano plane in Section 2.

Theorem 3. *Let P be a set of eight players. Then, any stable SKC $\mathcal{T}$ on P is equivalent to $\mathcal{T}_F$.*

Proof. Without loss of generality, $P = \{0,1,2,3,4,5,6,7\}$, and we put $P^* := P \setminus \{0\}$. Our argument will have three parts. First, we show that any stable SKC $\mathcal{T}$ on P gives rise to a subset $\ell_p \subseteq P^*$ for each $p \in P^*$, and we show the following properties:

$$p \in \ell_p \text{ for all } p \in P^*, \tag{2}$$

$$|\ell_p| = 3 \text{ for all } p \in P^*, \tag{3}$$

$$\#\{q \in P^* : p \in \ell_q\} = 3 \text{ for all } p \in P^*, \tag{4}$$

$$|\ell_p \cap \ell_q| = 1 \text{ for any distinct } p, q \in P^*. \tag{5}$$

Table 5. Stable SKC on 16 players.

Knockout tournament	Seeding
1	0123-4567-89ab-cdef
2	0246-8ace-3175-b9fd
3	0365-cfa9-b8de-7412
4	048c-37bf-62ea-51d9
5	05af-72d8-eb41-9c36
6	06ca-bd71-539f-e824
7	07e9-f816-da34-25cb
8	083b-6e5d-c4f7-a291
9	0918-2b3a-4d5c-6f7e
10	0a7d-e493-f582-1b6c
11	0b5e-a1f4-7c29-d683
12	0cb7-59e2-a61d-f348
13	0d94-1c85-2fb6-3da7
14	0ef1-d32b-9768-4ab5
15	0fd2-964b-1ec3-875a

Table 6. Stable SKC on 32 players.

Knockout tournament	Seeding
1	0123-4567-89ab-cdef-ghij-klmn-opqr-stuv
2	0246-8ace-gikm-oqsu-5713-df9b-lnhj-tvpr
3	0365-cfa9-orut-knih-lmjg-pqvs-deb8-1274
4	048c-gkos-51d9-lhtp-ae26-quim-fb73-vrnj
5	05af-khur-d872-psjm-qvgl-eb41-nito-369c
6	06ca-ouki-ljpv-db17-f935-nhrt-qsmg-24e8
7	07e9-sril-tqjk-16f8-vohm-34 da-25cb-upgn
8	08go-5dlt-a2qi-f7vn-ks4c-hp19-ume6-rjb3
9	09ir-18jq-2bgp-3aho-4dmv-5cnu-6fkt-7els
10	0aku-d7pj-qge4-nt39-hr5f-sm82-b1vl-6cio
11	0bmt-92vk-ip4f-rgd6-1ans-83ul-jo5e-qhc7
12	0cok-lpd1-f3nr-qm2e-ui6a-b7jv-ht95-48sg
13	0dqn-hsb6-7atg-mrc1-e3kp-vi58-94ju-ol2f
14	0esi-tj1f-vh3d-2cug-rl79-68qk-4aom-pn5b
15	0fuh-pm78-no96-e1gv-b4lq-itc3-sj2d-5ark
16	0g5l-aqfv-k4h1-uerb-dt8o-7n2i-p9sc-j3m6
17	0h7m-ev9o-sdra-i3l4-tcqb-j2k5-1g6n-fu8p
18	0i1j-2g3h-4m5n-6k7l-8q9r-aobp-cudv-esft
19	0j3g-6l5m-cvfs-ap9q-obr8-udte-k7n4-i1h2
20	0kdp-qen3-h5s8-bv6i-7jau-t9g4-m2rf-co1l
21	0lfq-ubh4-pcm3-7i8t-n2od-9s6j-er1k-g5va
22	0m9v-i4rd-1n8u-j5qc-2kbt-g6pf-3las-h7oe
23	0nbs-m1ta-9u2l-v8k3-i5pe-4jfo-rcg7-dq6h
24	0old-fnq2-u6bj-h94s-p1ck-me3r-7via-8gt5
25	0pne-bis5-mf1o-t4aj-9gu7-2rlc-v68h-kd3q
26	0qhb-7tmc-ekv5-9jo2-s6dn-r1ag-i83p-lf4u
27	0rj8-3ogb-6tle-5umd-cnv4-fks7-ahp2-9iq1
28	0st1-v32u-r76q-4op5-jfei-cghd-8kl9-nbam
29	0tv2-r64p-jech-8lna-3us1-o57q-gdfi-bmk9
30	0up7-n9eg-blic-s25r-m8fh-1vo6-t34q-akjd
31	0vr4-jc8n-3so7-gfbk-6pt2-laeh-5qu1-m9di

Next, we argue that the configuration $\{(p, \ell_p) : p \in P^*\}$, in turn, uniquely determines the stable SKC $\mathcal{T}$ that gave rise to it; that is, if two stable SKCs $\mathcal{T}$ and $\mathcal{T}'$ on P give rise to the same configurations, then $\mathcal{T} = \mathcal{T}'$. Finally, we observe that if sets ℓ_p satisfy these properties, then viewed geometrically as lines through points P^*, they form a Fano plane as in Figure 3, after applying a suitable permutation s of P^* to the point labels. Moreover, $\mathcal{T}_F$ exactly gives rise to the Fano plane as in Figure 3. Extending s to a bijection $s : P \to P$ by setting $s(0) = 0$, it then follows that $\mathcal{T}_F(s) = \mathcal{T}$.

So, let $\mathcal{T}$ be a stable SKC on P. For each $p \in P^*$, there is a unique $T \in \mathcal{T}$ so that $v_T(0, p) = 1$, and we will call this tournament T_p. Then, $\mathcal{T} = \{T_p : p \in P^*\}$. For each $p \in P^*$, let

$$\ell_p := \{q \in P^* : v_{T_p}(0, q) \leq 2\}.$$

Properties (2) and (3) are evident from these definitions. Property (4) follows because

$$\{q \in P^* : p \in \ell_q\} = \{T \in \mathcal{T} : v_T(0, p) \leq 2\}$$

and $\#\{T \in \mathcal{T} : v_T(0, p) = i\} = 2^{i-1}$ for $i = 1, 2$ as $\mathcal{T}$ is stable. To see (5), suppose for a contradiction that $r, s \in \ell_p \cap \ell_q$ and $r \neq s$. Then,

$$\mathcal{S} := \{T \in \mathcal{T} : v_T(0, r) \leq 2 \text{ and } v_T(0, s) \leq 2\} \supseteq \{T_p, T_q\}.$$

Note that $v_T(r, s) = 3$ if and only if exactly one of $v_T(0, r) \leq 2$ and $v_T(0, s) \leq 2$ holds true. Then,

$$\#\{T \in \mathcal{T} : v_T(r, s) = 3\} = \#\{T \in \mathcal{T} : v_T(0, r) \leq 2 \text{ or } v_T(0, s) \leq 2\} - \#\mathcal{S} =$$

$$\#\{T \in \mathcal{T} : v_T(0, r) \leq 2\} + \#\{T \in \mathcal{T} : v_T(0, s) \leq 2\} - 2\#\mathcal{S} \leq 3 + 3 - 2 \cdot 2 = 2.$$

This contradicts that $\#\{T \in \mathcal{T} : v_T(r, s) = 3\} = 4$ as $\mathcal{T}$ is stable.

We next show that the configuration $\{(p, \ell_p) : p \in P^*\}$, in turn, uniquely determines the stable SKC $\mathcal{T}$. Precisely, we show that for each $p \in P^*$, the tournament T_p is uniquely determined by the relative position of p, ℓ_p in the configuration using only the definition of the lines; Properties (2), (3), (4), and (5); and the fact that $\mathcal{T}$ is stable. By the definition of ℓ_p in terms of T_p, we have $v_{T_p}(q,r) = 3$ if and only if $\ell_p \cup \{0\}$ contains exactly one of q, r. Because the remaining pairs $\{q, r\}$ have $v_{T_p}(q,r) \in \{1,2\}$, it suffices to show that the four pairs $\{q, r\}$ so that $v_{T_p}(q,r) = 1$ are fixed by the configuration. To this end, we show that $v_{T_p}(q,r) = 1$ if $\{q,r\} = \{0,p\}$ or if $\{q,r\} = \ell_s \setminus \{p\}$ for some $s \in P^*$. It is clear that $v_{T_p}(0,p) = 1$ and also, that $v_{T_p}(q,r) = 1$ if $\ell_p = \{p,q,r\}$. Suppose that $v_{T_p}(q,r) = 1$, but neither of the previous cases apply. Then, $\{q,r\} \subseteq P^* \setminus \ell_p$ and say, $\ell_p = \{p,t,u\}$. Consider the two lines ℓ_t, ℓ_u, and let $z \in \ell_t \cap \ell_u$. We have $\ell_z \cap \ell_p = \{p\}$ because ℓ_z must intersect each of ℓ_p, ℓ_t, ℓ_u once. Naming the remaining elements of P^*, we have $\ell_t = \{t,w,z\}, \ell_u = \{u,x,z\}$, and $\ell_z = \{p,y,z\}$. Because p is on three lines that only intersect in p, the set $\{p, w, x\}$ is also a line, and we may assume that $\{p,w,x\} = \ell_w$ (possibly after switching t with u and switching w with x). There are three ways in which the four elements $P^* \setminus \ell_p = \{w,x,y,z\}$ can be split up in two pairs $\{q, r\}$ so that $v_{T_p}(q,r) = 1$. If $v_{T_p}(y,z) = v_{T_p}(w,x) = 1$, then we are done because $\{y,z\} = \ell_z \setminus \{p\}$ and $\{w,x\} = \ell_w \setminus \{p\}$. If not, then $v_{T_p}(w,z) = v_{T_p}(x,y) = 1$ or $v_{T_p}(x,z) = v_{T_p}(w,y) = 1$. Either possibility would contradict that $\mathcal{T}$ is stable: in the former case because $\{w,z\} = \ell_t \setminus \{t\}$ and hence, $v_{T_t}(w,z) = 1$ and in the latter case because $\{x,z\} = \ell_u \setminus \{u\}$ and hence, $v_{T_u}(x,z) = 1$.

Finally, we show that the configuration $\{(p,\ell_p) : p \in P^*\}$ that arises from $\mathcal{T}$ equals the Fano plane of Figure 3 with lines associated with points as in Table 1, up to a permutation of the points P^*. Indeed, following the analysis of uniqueness (starting from any fixed $p \in P^*$), there is just one way to choose the remaining two lines, namely $\ell_y = \{y,w,u\}$ and $\ell_x = \{t,x,y\}$. Fix a permutation s of P by setting

$$s(0) = 0,\ s(1) = p,\ s(2) = z,\ s(3) = y,\ s(4) = t,\ s(5) = u,\ s(6) = w,\ s(7) = x.$$

Then, replacing each point label in Figure 3 with its image under s yields the configuration that arises from $\mathcal{T}$. Moreover, a straightforward check shows that the Fano plane of Figure 3 is the configuration that arises from $\mathcal{T}_F$; indeed, our construction of the lines ℓ_p is just a reversal of the construction used to create $\mathcal{T}_F$ from the Fano plane. Then, $\mathcal{T} = \mathcal{T}_F(s)$. □

For $n = 2^k > 8$ players, we do not know whether there exist stable SKCs that are inequivalent to the ones that we construct in this paper, not even in the case of $n = 16$ players.

6. Balancing the Order of Play in a Stable SKC

In this section, we focus on a property that is relevant for the application sketched in Section 1.1; its relevance has been outlined to us by Fairhurst [4]. In the Premier League of Darts, the seven games of each knockout tournament are played sequentially on a match day. In particular, the four first-round matches are played before the semifinals and the final. These four matches are played in so-called *slots*; the first match is played in slot 1, the second match is played in slot 2, the third match is played in slot 3, and the last first-round match is played in slot 4. As an illustration, let $s = 0123\text{-}4567$ be the seeding of a knockout tournament of eight players. We will assume that the order of play of the first-round matches prescribed by s can be found by viewing s from left to right. Then, the order of play that follows has player 0 versus player 1 in slot 1, player 2 versus player 3 in slot 2, player 4 versus player 5 in slot 3, and player 6 versus player 7 in slot 4.

The PDC prefers to balance, across the SKC, the occurrence of each player in the first-round matches over these four slots. The motivation for this wish stems from aiming to have, on average, equal rest times for the players before the semifinals.

Because in general, an SKC consists of $2^k - 1$ knockout tournaments and because a seeding s has 2^k positions, it is impossible to select $2^k - 1$ seedings such that each player occurs exactly once in each position of the seeding s.

However, one may wonder whether it is possible to choose $2^k - 1$ seedings s such that for any pair of seedings s, s' and position j, we have $s_j \neq s'_j$. This would be of great interest for the Premier League of Darts as they want the players as evenly distributed over the slots as possible during the competition.

Notice that in the construction of a stable SKC, we have chosen all seedings such that $s_0 = 0$. As can be seen in Table 4–6 and as can easily verified in general, for all other indices $j \neq 0$, it holds that $s_j \neq s_{j'}$. Except for player 0, who in these SKCs, is always scheduled to play in the first slot, all other players are scheduled at a different position of the seedings, and hence, each other player occurs once in the first slot and twice in each other slot of the SKC. Therefore, it is a relevant question to what extent it is possible to spread *all* players as good as possible over the slots of an SKC that is stable. We prove the following result about a stable SKC of eight players (which is the Premier League format).

Theorem 4. *Let $\mathcal{T} = \{T_1, \ldots, T_7\}$ be a stable SKC of eight players. Then, there is a player who across all seven knockout tournaments, is scheduled to play in at most three of the four slots.*

Proof. The SKC consists of seven knockout tournaments, so there exists a player who plays at least four times in the first two slots. Without loss of generality, this is player 0. We distinguish three cases.

Case 1. If player 0 is scheduled to play six or seven times in one of the first two slots, we are done as then, player 0 will not play in either slot 3 or slot 4.

Case 2. Suppose player 0 is scheduled to play four times in slots 1, 2. Clearly, then player 0 plays three times in slots 3, 4. We know from Theorem 3 that the seedings in $\mathcal{T}$ can be constructed using the Fano plane and its node-line pairs. There are three node-line pairs (x, ℓ) that construct a round in $\mathcal{T}$ where player 0 does not play in the slots 1, 2. The properties of the Fano plane imply that either there must be a player $z \neq 0$ who is on all three lines in these node-line pairs or there is a player $z \neq 0$ who is on neither of these three lines.

In the first subcase, this player z is scheduled in slots 3, 4 during these three rounds (this follows from the fact that all three players on a line are scheduled in the same half as player 0 and play in the half opposite to player 0 in the other four rounds—when player 0 is scheduled in slots 1, 2). Hence, in all seven rounds, player z is scheduled in slots 3, 4 and is never scheduled in slots 1, 2.

In the latter case, player z is scheduled in slots 1, 2 during these three rounds. Stability of the SKC implies that player z is scheduled in the same half as player 0 exactly three times. All these times can only occur during the remaining four rounds when player 0 is scheduled in slots 1, 2. Thus, player z is scheduled to play six of seven rounds in slots 1, 2 and scheduled to play only one time in slots 3, 4.

Case 3. Suppose that player 0 is scheduled to play five times in slots 1, 2. Again, we use that all rounds can be constructed from a node-line pair. Two lines in the Fano plane always have one point in common; thus, there must be a player z who plays in the same half as player 0 during the two rounds when player 0 is scheduled in slots 3, 4. In the other five rounds, this player z will play in the half opposite to player 0 four of five times, implying that player z will be scheduled in round 3 and round 4. In total, player z will be scheduled six times in slots 3, 4 and will be scheduled only one time in slots 1, 2.

In all these cases, there is a player who is scheduled to play at most once in either slots 1, 2 or slots 3, 4 combined—so never in both slots. This finishes the proof. □

7. Discussion

We have analyzed a novel tournament design that is used in practice and that can be seen as a combination of a knockout tournament and a round-robin tournament; we call it a serial knockout competition. From the viewpoint of fairness, an attractive property of an SKC is stability: whether pairs of players can meet equally often in the rounds of the SKC. We have shown that this is always possible. We have also shown that the construction to create stable SKCs generates a unique (up to permutation of the players) tournament for eight players; it remains open to generalize this for larger powers of two.

References

[1] Aziz H, Gaspers S, Mackenzie S, Mattei N, Stursberg P, Walsh T (2014) Fixing a balanced knockout tournament. Woolridge M, Dy J, Natarajan S, eds. *Proc. Twenty Eighth AAAI Conf. Artificial Intelligence* (AAAI Press, Palo Alto, CA), 552–558.
[2] Chavez A, O'Neill C (2022) The fundamental theorem of finite fields: A proof from first principles. *Amer. Math. Monthly* 129(3):268–275.
[3] Considine J, Gallagher L (2018) Competitive balance in a quasi-double knockout tournament. *Appl. Econom.* 50(18):2048–2055.
[4] Fairhurst G (2023) Data via personal communication with the authors, Communication between July and October 2021, per mail.
[5] Fayers M (2005) Multiple-elimination knockout tournaments with the fixed-win property. *Discrete Math.* 290(1):89–97.
[6] Groh C, Moldovanu B, Sela A, Sunde U (2012) Optimal seedings in elimination tournaments. *Econom. Theory* 49(1):59–80.
[7] Horen J, Riezman R (1985) Comparing draws for single elimination tournaments. *Oper. Res.* 33(2):249–262.
[8] Karpov A (2016) A new knockout tournament seeding method and its axiomatic justification. *Oper. Res. Lett.* 44(6):706–711.
[9] Manurangsi P, Suksompong W (2022) Fixing knockout tournaments with seeds. Preprint, submitted April 24, https://arxiv.org/pdf/2204.11171.pdf.
[10] Vu T (2010) Knockout tournament design: A computational approach. PhD thesis, Stanford University, Stanford, CA.
[11] Vu T, Shoham Y (2011) Fair seeding in knockout tournaments. *ACM Trans. Intelligent Systems Tech.* 3:9.
[12] Wikipedia (2022) Premier league darts 2022. Accessed May 1, 2022, https://nl.wikipedia.org/wiki/Premier_League_Darts_2022.