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Algorithms for Recognizing Economic Properties in Matrix Bid Combinatorial Auctions

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A combinatorial auction is an auction where multiple items are for sale simultaneously to a set of buyers. Furthermore, buyers are allowed to place bids on subsets of the available items. This paper focuses on a combinatorial auction where a bidder can express his preferences by means of a so-called *ordered matrix bid*. Ordered matrix bids are a bidding language that allows a compact representation of a bidder's preferences and was developed by Day [Day, R. W. 2004. Expressing preferences with price-vector agents in combinatorial auctions. Ph.D. thesis, University of Maryland, College Park]. We give an overview of how a combinatorial auction with matrix bids works. We discuss the relevance of recognizing whether a given matrix bid has properties related to elements of economic theory such as free disposal, subadditivity, submodularity, and the gross substitutes property. We show that verifying whether a matrix bid has these properties can be done in polynomial time by solving one or more shortest-path problems. Finally, we investigate to what extent randomly generated matrix bids satisfy these properties.

Key words: combinatorial auction; matrix bids; free disposal; subadditivity; submodularity; gross substitutes; expressiveness
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1. Introduction

A combinatorial auction is an auction where multiple items are for sale simultaneously to a set of buyers. In a combinatorial auction a buyer is allowed to place bids on subsets of the items. These subsets are sometimes called *bundles*. The auctioneer decides—after one or more rounds or after a certain amount of time, depending upon the design—to accept some of the bids and to allocate the items to the bidders accordingly. It may occur that a bidder values a set of items higher than the sum of his valuations for the individual items of this set. If this is the case, we say that these items are complements to this bidder; if the converse is true we say these items are substitutes. Notice that these complementarity or substitution effects may be bidder specific. The main advantage of a combinatorial auction is that it allows a bidder to express his preferences to a greater extent. (His can be replaced by her, and he by she.)

The popularity of combinatorial auctions (and corresponding research) has increased in recent years. As a result, there are many examples where combinatorial auctions prove to be a successful way to market items. The airline sector offers several possibilities for combinatorial auctions: landing slots (Ball et al. 2005, Rassenti et al. 1982), seats (Eso 2001), and flights (Bleischwitz and Kliewer 2005). We also find applications of combinatorial auctions in truckload transportation (Caplice and Sheffi 2005, Ledyard et al. 2002, Sheffi 2004) and allocating bus routes (Cantillon and Pesendorfer 2005). Finally, we mention some applications in the food industry: Epstein et al. (2002) use a combinatorial auction to assign catering contracts for meals in Chilean schools, and Mars, Inc., makes use of a combinatorial auction to arrange procurements with quantity discounts with its suppliers (Hohner et al. 2003).

In this paper, we investigate so-called *ordered matrix bids*, originally proposed by Day (2004). In a combinatorial auction in its most general form, bidders can bid whatever amount they please on any subset of items in which they are interested. This, however, immediately poses a problem when it comes to representing such a valuation: $2^m - 1$ numbers might be needed (where m is the number of items). Matrix bids combine a compact representation of a bidder's valuations with the possibility of bidding on any subset of the items. Indeed, a single matrix bid includes bids on any subset of the items, but requires only an ordered list of m items and $m(m+1)/2$ matrix bid entries. The downside is that a bidder generally cannot bid whatever amount he pleases for all bundles using a single matrix bid. However, when a bidder uses multiple matrix bids, the resulting bidding language is as expressive as the most sophisticated bidding languages in the literature (Day and Raghavan 2009).

Bids in any practical combinatorial auction are likely to possess some structure. In the literature, we find references of both theoretical structures (see, e.g., Rothkopf et al. 1998, Nisan 2000, Leyton-Brown and Shoham 2005) and structures in practice (see, e.g., Bleischwitz and Kliewer 2005, Goossens et al. 2007). Capturing and understanding this structure is important since it improves our understanding of various properties of an auction. Furthermore, it allows us to develop algorithms that can be more efficient than algorithms for a general combinatorial auction.

One of the most challenging problems related to combinatorial auctions is to decide which bidders should get what items. This problem is called the *winner determination problem*. Sandholm (2002) shows that the winner determination problem cannot be approximated within a ratio of $\max(K^{\epsilon-1}, m^{\epsilon-1/2})$ in polynomial time for any fixed $\epsilon > 0$ (unless $P = ZPP$), where K is the number of bundles on which a bid has been made. The matrix bid auction allows for a faster computation because of the restriction on the preferences that is assumed. Despite this restriction, the winner determination problem of the matrix bid auction is NP-hard (Day 2004). In addition, there exists no polynomial-time approximation scheme for the winner determination problem for the matrix bid auction even when all bidders have an identical ranking of the items, unless $P = NP$ (Goossens and Spieksma 2009). Nevertheless, Day and Raghavan (2009) solve the winner determination problem of the matrix bid auction using a formulation based on the assignment problem, and compare this with solving the winner determination problem of a general combinatorial auction using a formulation based on the set-packing problem. The authors conclude that the computation time for the general combinatorial auction is higher and grows much faster than for the matrix bid auction. Moreover, they manage to solve the winner determination problem for matrix bid auctions with 72 items, 75 bidders, and over 10^{23} bids, whereas for the general combinatorial auction, the largest instances that can be solved have 16 items, 25 bidders, and fewer than 10^9 bids. Although this comparison is somewhat distorted because it does not use a state-of-the-art method (e.g., CABOB; see Sandholm et al. 2005) to solve the winner determination problem for the general auction, it does give an indication of the size of the matrix bid auctions that can be solved in practice. Goossens and Spieksma (2009) developed two branch-and-price algorithms for the matrix bid auction, where the pricing problem is a shortest-path problem. These algorithms at least form a viable approach to solve instances of the matrix bid auction winner determination problem.

The core of this paper consists of recognizing whether a given matrix bid satisfies properties related to economic theory such as free disposal, subadditivity, submodularity, and the gross substitutes property. It is interesting to know whether bids have these properties, because it can allow the use of faster algorithms to solve the winner determination problem exactly or result in better approximation results. For instance, if the bids are submodular (see §3.3), a greedy algorithm produces a 1/2-approximation for this problem (Lehmann et al. 2006). If the bids satisfy the gross substitutes property (see §3.4), the winner determination problem can be solved in polynomial time, and the LP-relaxation of a set-packing formulation results in an integral solution (Kelso and Crawford 1982, Bikhchandani et al. 2002). Furthermore, knowledge about the bids can result in some insights on the optimal allocation. For instance, if the bids satisfy the free disposal property, there is an allocation that maximizes the total winning bid value, in which all items are awarded to some bidder (see §3.1). If each bidder's bids satisfy the gross substitutes property, then there exists an allocation and prices that form a Walrasian equilibrium (see §3.4 and Nisan and Segal 2006).

The rest of this paper is organized as follows. In §2, we give an overview of how an auction with ordered matrix bids works. In §3, we discuss free disposal, subadditivity, submodularity, and the gross substitutes property, and we show how to verify in polynomial time whether a matrix bid has any of these properties. Indeed, we show that the recognition of these properties can be done by solving one or more shortest-path problems. Finally, in §4, we implement these results to investigate to what extent randomly generated matrix bids satisfy these economic properties.

2. The Matrix Bid Combinatorial Auction

The matrix bid combinatorial auction (for short, matrix bid auction) is a multi-item, single-unit combinatorial auction. This means that for each item that is auctioned, only one unit of this item is available.

In the matrix bid auction, each bidder must submit a strict ordering (or ranking) of the items in which he is interested. Furthermore, we assume that for each bidder, the extra value an item adds to a set is determined only by the number of higher-ranked items in that set, according to the ranking of that bidder.

Let G be the set of items, indexed by i, and let B be the set of bidders, indexed by j. The ordering of the items is denoted by r_{ij}, which is item i's position in bidder j's ranking, for each $i \in G$ and $j \in B$. This ordering should be strict in the sense that for each bidder j, $r_{i_1 j} \neq r_{i_2 j}$ for any pair of distinct items i_1 and i_2. For instance, if $r_{ij} = 2$, item i is bidder j's second-highest ranked item. Furthermore, each bidder j specifies values b_{ijk}, which correspond to the value the bidder is prepared to pay for item i given that it is the kth highest ranked item in the set that bidder j is awarded. The b_{ijk} values allow us to determine the value bidder j attributes to any set $S \subseteq G$. Indeed, bidder j's bid on a set S is denoted as $b_j(S)$ and can be computed as

$$b_j(S) = \sum_{i \in S} b_{i,\,j,\,k(i,\,j,\,S)}, \tag{1}$$

where $k(i, j, S)$ is the ranking of item i among the items in the set S, according to bidder j's ranking. Notice that Equation (1) assumes that no externalities are involved; i.e., a bidder's valuation depends only on the items he wins, and not, for instance, on the identity of the bidders to whom the other items are allocated. The winner determination problem is, given the bids $b_j(S)$ for each set S and each bidder j, to determine which bidder is to receive which items such that the total winning bid value is maximized. Notice that we assume that each bidder pays what he bids for the subsets he wins.

Observe that the value for index k of item i in bidder j's bid can never be higher than the rank r_{ij}. This allows us to arrange the values b_{ijk} as a lower triangular matrix for each bidder j, where the rows correspond to the items, ordered by decreasing rank, and the columns correspond to values for k, hence, the name matrix bid (with order). Notice also that bidder j's ranking r_{ij} does not necessarily reflect a preference order of the items. If an item is highly ranked, this merely means that its added value to a set depends on fewer items than the added value of a lower-ranked item. Furthermore, we make no assumption regarding the b_{ijk} values. Indeed, these values may be negative, e.g., to reflect the disposal cost of an unwanted item. Specifying a sufficiently large negative value can also keep the bidder from winning this item at all. We assume that bids are normalized, in the sense that any bidder's bid on the empty set is zero.

As a frivolous example, we consider the following matrix bid, where a bidder expresses his preferences for an ice cream (ice). There are two flavors of ice cream (vanilla and banana); hot chocolate and strawberry sauce also are also available.

Vanilla ice	4			
Banana ice	5	2		
Hot chocolate	−5	0	3	
Strawberry sauce	−5	0	3	−1

Consider now the value this bidder j attributes to vanilla ice with hot chocolate. Observe that for this choice of S, vanilla ice is the highest-ranked item (that is, $k(\text{vanilla ice}, j, S) = 1$) and hot chocolate is the second-highest ranked item (that is, $k(\text{hot chocolate}, j, S) = 2$). We find using (1)

$$\begin{aligned} b_j(S) &= b_{\text{vanilla ice},\,j,\,k(\text{vanilla ice},\,j,\,S)} \\ &\quad + b_{\text{hot chocolate},\,j,\,k(\text{hot chocolate},\,j,\,S)} \\ &= b_{1,\,j,\,1} + b_{3,\,j,\,2} \\ &= 4 + 0 = 4. \end{aligned}$$

Thus, this matrix bid can be interpreted as follows: bidder j feels that he needs at least one scoop of ice cream of one of the two available flavors, although he prefers banana. Indeed, no combination without ice cream will result in a positive valuation, because the bidder charges a (disposal) cost of five if he gets one or both toppings without ice cream. Furthermore, the bidder is not willing to pay as much for the second scoop of ice cream as for the first. The highest bid this bidder places is nine, for the combination of vanilla and banana ice with either of the two toppings.

The matrix bid auction offers a way of capturing structure that may be present in combinatorial auctions. Day (2004) illustrates that this structure encompasses at least the following six known types of bidders.

Additive preference bidder: For every item i, the bidder has a price p_i. The bidder's valuation for a set S is then $\sum_{i \in S} p_i$.

Single-minded bidder: This bidder is interested in one particular set S for which he is willing to pay a price p. These single-minded bids (S, p) are also known as flat bids or atomic bids (Nisan 2000).

Nested flat bidder: This bidder is a generalization of the single-minded bidder and expresses a chain of q exclusive single-minded bids (S_1, p_1), $(S_2, p_2), \ldots, (S_q, p_q)$ such that $S_1 \subset S_2 \subset \cdots \subset S_q$.

Nested k-of bidder: The k-of bid function consists of a bid (k, S, p), which is a bid of p on any subset of S of at least size k. Multiple k-of bids $(k_1, S, p_1), (k_2, S, p_2), \ldots, (k_q, S, p_q)$ on the same set S can be represented in a single matrix bid, provided that all k-values are different. This bid function is also known as the general symmetric bid function, in which the bidder specifies prices $p_1, p_2, \ldots, p_m$, where

p_k is the price the bidder is willing to pay for the kth item won (see Nisan 2000). The bidder's valuation for a set S is then $\sum_{i=1}^{|S|} p_i$.

Partition bidder: This bidder partitions the items into a number of groups of substitutes. The bidder gives a ranking of the groups and prices he is willing to pay for receiving exactly one item from each group, given that exactly one item from each higher-ranked group has been received. This bid function can be generalized to accommodate an arbitrary given demand for each group of substitutes.

Add-on bidder: This bid function consists of a bid for an essential item and extra prices the bidder is willing to pay for each number of items from a set of add-on items in which the bidder is interested.

Any auction whose bidders are from these types can be represented as a combinatorial auction with a single matrix bid per bidder.

3. Recognizing Matrix Bid Properties

In this section, we discuss the relationship between the bid function implied by a matrix bid and economic concepts as free disposal (§3.1), complement freeness (§3.2), decreasing marginal valuations (§3.3), and the gross substitutes property (§3.4). In particular, we show how to verify efficiently whether a given matrix bid satisfies each of these properties. Because this section deals only with the bid function of a single bidder, the index j that is used to indicate the bidder will be dropped. In the literature, many of the economic concepts discussed in this section are in terms of a valuation function. Although for some auctions (e.g., the Vickrey-Clarke-Groves auction) it has been shown that it is in the bidder's best interest to bid his true valuation (see Cramton et al. 2005), in general, a bidder's bid and his valuation need not be identical. Indeed, strategic considerations may motivate a bidder to express bids that differ considerably from his valuation. Nevertheless, in this section, we ignore this issue and assume equivalence between the notions bid function and valuation function. This is common practice in studies on bidding languages (see, e.g., Nisan 2005). We refer to Gul and Stacchetti (1999) and Cramton et al. (2005) for the definitions used in this section.

3.1. Free Disposal

In economics, it is often assumed that bidders prefer more to less. In the context of an auction, this means that a bidder is always willing to receive one or more items for free. Consequently, a seller will never get stuck with unsold items and can therefore dispose of any number of items at no cost. Free disposal is a very common assumption in the literature on combinatorial auctions (Lehmann et al. 2006, Nisan 2000) and can be defined as follows.

Definition 1. A bid function b satisfies the free disposal property if and only if

$$b(S) \le b(T) \quad \forall\, S \subseteq T \subseteq G. \tag{2}$$

Notice that this definition is equivalent to the definition of a monotone nondecreasing function. Alternatively stated, this definition implies that disposing an item from a set cannot increase the value of the resulting set.

In combinatorial auctions where a bidder does not communicate bids on every possible subset of items but rather only on a limited number of subsets of his liking, allocating all items may be problematic for the auctioneer. In its strictest sense, the lack of free disposal would mean that buyers do not accept anything extra beyond what they bid on. Using this interpretation, even finding a solution for the winner determination problem where all items are allocated is NP-complete (Sandholm et al. 2002). However, many other approaches allow the auctioneer to allocate all items without disposal cost. Nisan (2005) assumes that bids of each bidder satisfy the free disposal property. Moreover, if a bidder did not express a bid on a set S, the auctioneer can construct a new bid $b(S)$ equal to the highest bid over all subsets of S. Obviously, the newly created bids also satisfy the free disposal property. A similar approach is followed by Leyton-Brown et al. (2000) because they allow the auctioneer to create additional bids with a value of zero for any subset of items, which can then be combined with any of the bids (which also satisfy the free disposal property) expressed by the bidders.

The concept of free disposal is also relevant in a combinatorial auction where bidders do express bids on every possible subset, as in the matrix bid auction. In this case, assuming bid functions that satisfy free disposal guarantees the existence of a total winning bid value maximizing allocation in which all items are awarded to some bidder.

Obviously, not every matrix bid satisfies the free disposal property. For instance, the matrix bid that was used in §2 does not allow free disposal, because $b(\text{hot chocolate}) < b(\varnothing)$. The matrix framework indeed allows the bidder to take into account a disposal cost that can vary across the items and may lead to one or more sets with a negative valuation. However, imposing that each entry in the matrix bid is nonnegative is not sufficient to attain the free disposal property. This is illustrated by the matrix bid below, because $b(y) = 3 > 2 = b(x, y)$.

Item x	1	
Item y	3	1

However, having nonnegative entries that are nondecreasing in the rows is a sufficient condition for free

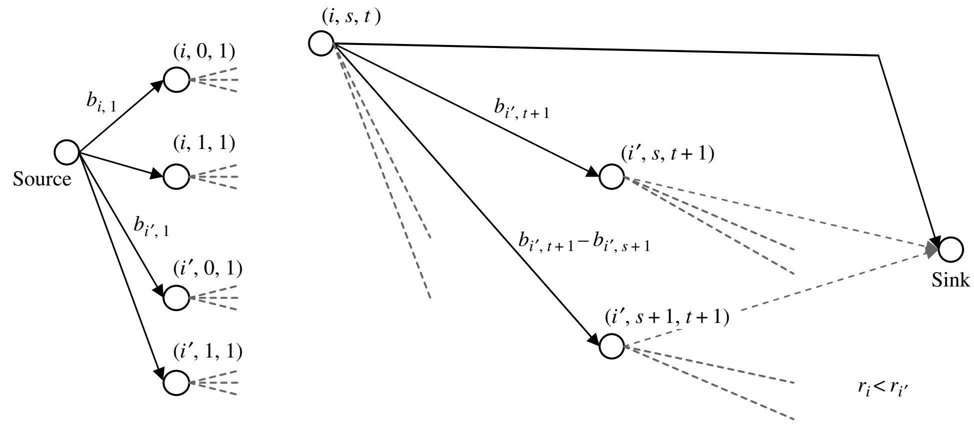

Figure 1 **Graph Used to Verify Free Disposal**

disposal. Indeed, in this case, each additional item will result in greater valuations for those items ranked lower. Furthermore, having nonnegative entries that are nonincreasing in the columns is also a sufficient condition. Indeed, by introducing an extra item in a set, an extra column (containing only nonnegative entries) will be used to determine the valuation of the new set, and from the columns that were already used, entries corresponding to a higher row, which are not lower than the ones originally used, will determine the valuation of the set. Thus, the valuation cannot decrease when an extra item is added to a set. Although it seems somewhat counterintuitive, this also means that matrix bids with nonnegative entries that are nonincreasing in the rows will satisfy free disposal if these entries are also nonincreasing in the columns. The matrix bid below satisfies free disposal and shows that having nonnegative entries that are nondecreasing in the rows is not a necessary condition, and neither is having nonnegative entries that are nonincreasing in the columns.

Item x	1	
Item y	3	2

Verifying free disposal can be done in polynomial time for a given matrix bid, as witnessed by the following theorem.

Theorem 1. *Verifying whether a matrix bid b satisfies the free disposal property can be done in polynomial time.*

Proof. We will show that solving a shortest-path problem on an acyclic graph involving $O(m^3)$ nodes and $O(m^4)$ arcs determines whether a matrix bid b satisfies the free disposal property (2). The graph can be described as follows.

The graph contains a source and a sink, and nodes indexed by (i, s, t). The index i refers to item i and ranges from 1 to m. The index t ranges from 1 to r_i, whereas s ranges from 0 to t. There are arcs from each node (i, s, t) to $(i', s, t+1)$ and to $(i', s+1, t+1)$ for all items i' ranked lower than i (recall that we consider a single bidder). Furthermore, there is an arc from the source to each node $(i, 0, 1)$ and $(i, 1, 1)$, and there is an arc from each node (except the source) to the sink. Let the cost on the arc from (i, s, t) to $(i', s, t+1)$ be equal to $b_{i', t+1}$, and let the cost on the arc from (i, s, t) to $(i', s+1, t+1)$ be equal to $b_{i', t+1} - b_{i', s+1}$. Analogously, the arcs from the source to each node $(i, 0, 1)$ and $(i, 1, 1)$ have a cost equal to $b_{i,1}$ and 0, respectively. Arcs to the sink have a cost equal to zero. This completes the description of the graph. Figure 1 illustrates this graph; arcs with no indication of their cost have a cost equal to zero.

The graph described above should be interpreted as follows. Each node (i, s, t) corresponds to a state where s and t items, ranked at least as high as item i, are present in set S and set $T \supseteq S$, respectively. Selecting an arc from (i, s, t) to $(i', s, t+1)$ corresponds to adding item i' to set T as the $(t+1)$th best item, but not to S, whereas an arc from (i, s, t) to $(i', s+1, t+1)$ corresponds to adding item i' to both set S and set T as the $(s+1)$th and $(t+1)$th best item, respectively. In this way, each path from source to sink determines sets S and T, and vice versa; there is a path from the source to the sink for each possible S and T. Notice that the arcs are such that S will always be a subset of T.

We now show that the length of a path from source to sink in this graph equals $b(T) - b(S)$. Each path from source to sink consists of two types of arcs: those arcs that add items i to the set T (and not to S) and those arcs that add items i to both S and T. The former arcs give rise to the term $\sum_{i \in T \setminus S} b_{i, k(i, T)}$, whereas the latter arcs give rise to the term $\sum_{i \in S \subseteq T} (b_{i, k(i, T)} - b_{i, k(i, S)})$. Recall from §2 that $k(i, A)$ denotes the rank of item i in the set A. Thus, the length of the path equals

$$\sum_{i\in T\setminus S} b_{i,k(i,T)} + \sum_{i\in S\subseteq T} (b_{i,k(i,T)} - b_{i,k(i,S)})$$
$$= \sum_{i\in T} b_{i,k(i,T)} - \sum_{i\in S} b_{i,k(i,S)}$$
$$= b(T) - b(S).$$

Thus, verifying the free disposal property (see Definition 1) for a given matrix bid can be done by solving a shortest-path problem in this graph. If the shortest path has a nonnegative length, then the matrix bid satisfies free disposal; otherwise, sets $S \subseteq T \subseteq G$ exist such that $b(S) > b(T)$, which means that the matrix bid does not have the free disposal property. Because the number of nodes ($O(m^3)$) and arcs ($O(m^4)$) of this acyclic graph is polynomial in the number of items, this takes polynomial time. □

3.2. Complement Freeness

Although the difficulty dealing with complementarity or substitution effects in a bidder's valuation in a classic sequential auction is a major motivation for researching combinatorial auctions in the first place, assuming the absence of complementarities or substitution effects is not uncommon in economic theory. Complement freeness can be especially relevant when multiple copies of the same item are desired, whereas substitute freeness could be more likely to occur when different economic resources are considered. (Lehmann et al. 2006, p. 271) state that "in most of microeconomic theory, consumers are assumed to exhibit diminishing marginal utilities." In their work, they assume that the valuation of a union of disjoint sets is never higher than the sum of the valuations of the individual sets. This notion can be formalized as follows and is also known as subadditivity.

Definition 2. A bid function b is complement free (or subadditive) if and only if

$$b(S\cup T) \le b(S) + b(T) \quad \forall S,\, T \subseteq G\colon S\cap T = \varnothing. \tag{3}$$

Although the winner determination problem for bidders with a complement-free bid function remains *NP*-hard (Lehmann et al. 2006), there exists a polynomial-time algorithm that finds a $O(1/\log m)$-approximation if, given external prices for all items, a bidder can determine in polynomial time for which set his valuation b exceeds the sum of the prices of the items in that set the most (Dobzinski et al. 2005). If a bidder can only determine his valuation for a given set in polynomial time, then the approximation ratio decreases to $O(m^{-1/2})$ (Dobzinski et al. 2005).

A sufficient condition to have a complement-free matrix bid is that the b_{ik} values are nonincreasing in the rows (i.e., $b_{ik} \ge b_{i,k+1}\ \forall i, k$). Indeed, for each item i, the valuation of the union of two sets will only make use of b_{ik} values with a value of index k at least as high as the value used in the valuation of the individual sets. Having nonincreasing b_{ik} values, however, is not a necessary condition, as is illustrated by the following example of a complement-free bid function.

Item x	0		
Item y	2	1	
Item z	2	1	2

We now show how we can verify in polynomial time whether a matrix bid is complement free.

Theorem 2. *Verifying whether a matrix bid b satisfies the complement-free property can be done in polynomial time.*

Proof. Given Definition 2, it suffices to establish the existence of sets of items S and T such that $S\cap T = \varnothing$ and $b(S) + b(T) - b(S\cup T) < 0$ to find out whether a matrix bid b is not complement free. We show that this can be done by solving a shortest-path problem by adapting the construction described in Theorem 1.

Consider a graph containing a source, a sink, and nodes, indexed by (i, s, t). The index i refers to item i, with $1 \le i \le m$. Both indices s and t range from 0 to r_i, insofar as $1 < s+t \le r_i$. There are arcs from each node (i, s, t) to $(i', s+1, t)$ and to $(i', s, t+1)$ for all items i' ranked lower than i. Furthermore, there is an arc from the source to each node $(i, 1, 0)$ and $(i, 0, 1)$, and there is an arc from each node (except the source) to the sink. Let the cost of the arc from (i, s, t) to $(i', s+1, t)$ be equal to $b_{i', s+1} - b_{i', s+t+1}$, and let the cost of the arc to $(i', s, t+1)$ be equal to $b_{i', t+1} - b_{i', s+t+1}$. The arcs from the source to each node $(i, 1, 0)$ and $(i, 0, 1)$ as well as all arcs to the sink have a cost equal to zero. Notice that this graph is acyclic. Figure 2 illustrates this graph; arcs with no indication of their cost have a cost equal to zero.

Each node (i, s, t) in the graph represents the state where s and t items, ranked at least as high as item i, are present in sets S and T, respectively. Each path from source to sink determines what items are to be added to sets S and T, and the arcs are such that these sets will always be disjoint. If such a path contains an arc from (i, s, t) to $(i', s+1, t)$, this corresponds to item i' being present in set S as the $(s+1)$th best item. If an arc from (i, s, t) to $(i', s, t+1)$ is present in the path, this means that item i' is in set T as the $(t+1)$th best item. We now show that the length of a path from source to sink in this graph equals $b(S) + b(T) - b(S\cup T)$. Each path from source to sink consists of two types of arcs: those arcs that add an item i to the set S and those arcs that add an item i to T. Given

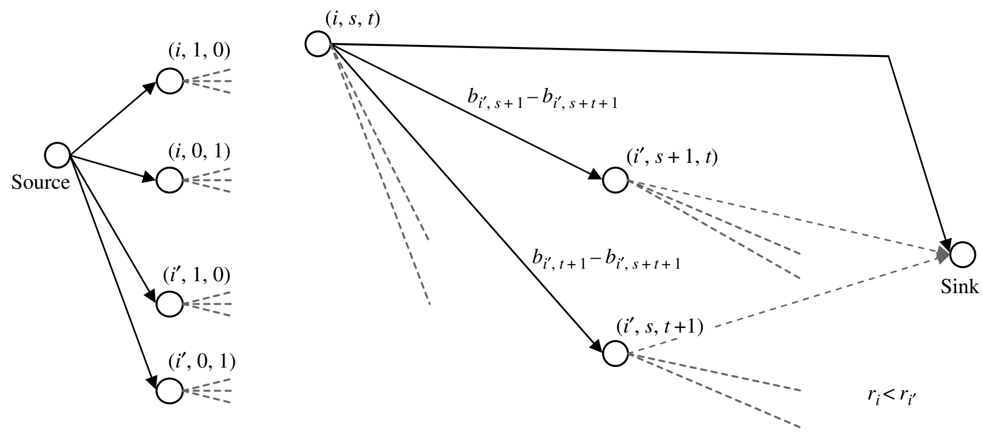

Figure 2 **Graph Used to Verify Complement Freeness**

the choice of the costs of the arcs, the length of a path equals

$$\begin{aligned}\sum_{i\in S}(b_{i,k(i,S)} - b_{i,k(i,S\cup T)}) + \sum_{i\in T}(b_{i,k(i,T)} - b_{i,k(i,S\cup T)})\\ = \sum_{i\in S} b_{i,k(i,S)} + \sum_{i\in T} b_{i,k(i,T)} - \sum_{i\in S\cup T} b_{i,k(i,S\cup T)}\\ = b(S) + b(T) - b(S\cup T).\end{aligned}$$

Thus, if a shortest path in this graph has a nonnegative length, then the matrix bid b is complement free, and vice versa. Because the graph is acyclic and contains a number of nodes and arcs that are polynomial in the number of items ($O(m^3)$ and $O(m^4)$, respectively), verifying whether a matrix bid is complement free can be done in polynomial time. □

Complement-free valuations find their natural counterpart in substitute-free valuations, for which $b(S \cup T) \geq b(S) + b(T)$ for all disjoint sets S and T. This property is also known as superadditivity. In this case, there is no need to prevent a bidder from winning multiple bids. Furthermore, the auctioneer only needs to take into account the highest bid for each set (although the winner determination problem for bidders with substitute-free valuations remains *NP*-hard).

Notice that having b_{ik} values that are nondecreasing in the rows is a sufficient condition for a matrix bid to be superadditive. Nondecreasing b_{ik} values in the rows is, however, not a necessary condition, which is illustrated by the matrix bid below.

Item x	0		
Item y	1	3	
Item z	2	4	3

Verifying whether a matrix bid satisfies the substitute-free property can also be done in polynomial time. Using the same graph as in Theorem 2, a longest path with nonpositive length implies superadditivity of the given matrix bid. Because this graph has no cycles, its longest path can be found in polynomial time.

Corollary 1. *Verifying whether a matrix bid b satisfies the substitute-free property can be done in polynomial time.*

3.3. Decreasing Marginal Valuations

In many practical applications, and also in most of economic theory, it is assumed that the more items a bidder has, the less he values an extra item. This concept is called decreasing marginal valuations.

Definition 3. A bid function b has decreasing marginal valuations if and only if

$$b(T\cup\{x\}) - b(T) \leq b(S\cup\{x\}) - b(S) \quad \forall S\subseteq T\subseteq G,\ x\in G. \quad (4)$$

It is well known (see, e.g., Moulin 1988) that a (bid) function has decreasing marginal valuations if and only if it is submodular.

Definition 4. A bid function b is submodular if and only if

$$b(S\cup T) + b(S\cap T) \leq b(S) + b(T) \qquad \forall S,\ T\subseteq G. \quad (5)$$

Lehmann et al. (2006) show that a valuation where the items have decreasing marginal valuations is also complement free (assuming that this valuation function satisfies free disposal and normalization). The authors also provide an example that illustrates that the converse is not true. Indeed, a complement-free valuation function may still have so-called hidden complementarities. When we consider a bidder's valuation for a set of items A, given that this bidder already acquired some set of items W ($W \cap A = \varnothing$), complementarities may still arise. In other words, even if a bid function b is complement free, this is not necessarily the case for marginal bids b', defined by $b'(A) = b(A \mid W) = b(A \cup W) - b(W)$. If we want to enforce that the marginal bids are complement free as well, the bid function b is required to be submodular (Lehmann et al. 2006).

Lehmann et al. (2006) also show that the winner determination problem for bidders with a submodular bid function remains *NP*-hard but that a greedy algorithm produces a 1/2-approximation. This algorithm simply assigns the items one by one (in no particular order) to the bidder with the highest marginal value for that item, given the other items that bidder already acquired. The following example with two items (x and y) and two bidders (A and B) shows that this approximation is tight, even for submodular matrix bids.

Bidder A			Bidder B		
Item x	1		Item x	1	
Item y	1	0	Item y	0	0

The optimal total winning bid value for the auctioneer in this auction is 2, by allocating item x to bidder B and item y to bidder A. However, the greedy algorithm can generate a total winning bid value of 1 by starting with allocating item x to bidder A and ending up with marginal bids of zero for item y. Khot et al. (2005) show that if each bidder can determine his valuation for a given set in polynomial time and if this valuation function is submodular, it is *NP*-hard to approximate the optimal solution by a factor better than $(e-1)/e$. This result assumes that, given external prices for all items, bidders cannot determine in polynomial time (in the number of items and bidders) for which set their valuation exceeds the sum of the prices of the items in that set the most.

Day (2004) suggests that a matrix bid with b_{ik} values that are nonincreasing in both the rows and the columns represents a submodular bid function. Notice from the example below that this is not necessarily the case (with $S = \{x, z\}$ and $T = \{y, z\}$).

Item x	7		
Item y	6	5	
Item z	5	1	0

Furthermore, not all bid functions with decreasing marginal valuations can be represented as a matrix bid, as can be easily verified for the submodular bid function b that produces the following bids on each subset of the item set $\{x, y, z\}$: $b(\{\}) = 0$, $b(\{x\}) = 1$, $b(\{y\}) = 2$, $b(\{z\}) = 3$, $b(\{x, y\}) = 3$, $b(\{y, z\}) = 3$, $b(\{x, z\}) = 3$, and $b(\{x, y, z\}) = 3$.

We can, however, verify whether a matrix bid represents a valuation function with decreasing marginal valuations in polynomial time.

Theorem 3. *Verifying whether a matrix bid b has decreasing marginal valuations can be done in polynomial time.*

Proof. Clearly, it is sufficient to establish the existence of sets of items S and T, for which $b(S) + b(T) - b(S \cup T) - b(S \cap T) < 0$, to find out whether a matrix bid is not submodular. We show that the existence of such sets can be verified by solving a shortest-path problem, again by adapting the construction described in Theorem 1.

Consider a graph containing a source, a sink, and nodes, indexed by (i, s, t, c). The index i refers to item i, with $1 \le i \le m$. The indices s, t, and c range from 0 to r_i, although no nodes are needed if both s and t are 0. There are arcs from each node (i, s, t, c) to $(i', s+1, t, c)$, $(i', s, t+1, c)$, and $(i', s+1, t+1, c+1)$ for all items i' ranked lower than i. Furthermore, there are arcs from the source to each node $(i, 1, 0, 0)$, $(i, 0, 1, 0)$, and $(i, 1, 1, 1)$, and there are arcs from each node except the source to the sink. Let the cost on the arc from (i, s, t, c) to $(i', s+1, t, c)$ be equal to $b_{i', s+1} - b_{i', s+t-c+1}$, and let the cost on the arc to $(i', s, t+1, c)$ be equal to $b_{i', t+1} - b_{i', s+t-c+1}$. The arcs from (i, s, t, c) to $(i', s+1, t+1, c+1)$ have a cost equal to $b_{i', s+1} + b_{i', t+1} - b_{i', s+t-c+1} - b_{i', c+1}$. The arcs from the source to each node $(i, 1, 0, 0)$, $(i, 0, 1, 0)$, and $(i, 1, 1, 1)$, and also all arcs to the sink have a cost equal to 0. Figure 3 illustrates this acyclic graph; arcs with no indication of their cost have a cost equal to zero.

The graph should be interpreted as follows. Each node (i, s, t, c) stands for a state where s, t, and c items ranked at least as high as item i that are in sets S, T, and $S \cap T$, respectively. Each path from source to sink determines what items are to be added to sets S and T, and there is a path from source to sink for each possible S and T. If the arc from (i, s, t, c) to $(i', s+1, t, c)$ is included in the path, this means that item i' is in set S as the $(s+1)$th best item, whereas the arc to $(i', s, t+1, c)$ corresponds to item i' being in set T, as the $(t+1)$th best item. The arc from (i, s, t, c) to $(i', s+1, t+1, c+1)$ corresponds to adding item i' to both S and T, where $c+1$ is the number of items in $S \cap T$ ranked at least as high as i'.

We now show that the length of a path from source to sink corresponds to $b(S) + b(T) - b(S \cup T) - b(S \cap T)$. Each path consists of three types of arcs: those arcs that add an item i to S and not to T, those that add an item i to T and not to S, and those that add an item i to both S and T. Given the choice of the costs of these arcs, the length of the path equals

$$\sum_{i \in S \setminus T} (b_{i, k(i, S)} - b_{i, k(i, S \cup T)}) + \sum_{i \in T \setminus S} (b_{i, k(i, T)} - b_{i, k(i, S \cup T)}) + \sum_{i \in S \cap T} (b_{i, k(i, S)} + b_{i, k(i, T)} - b_{i, k(i, S \cup T)} - b_{i, k(i, S \cap T)})$$

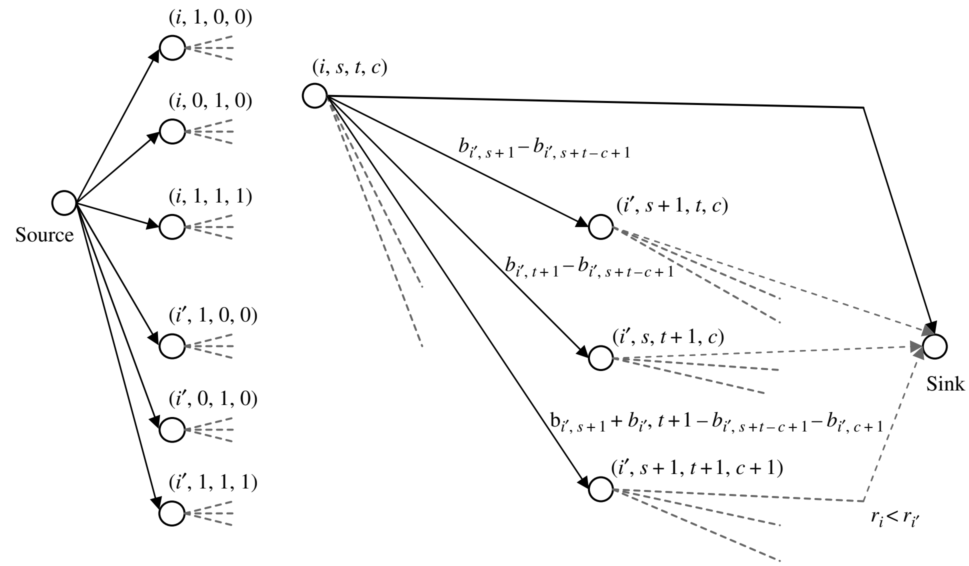

Figure 3 **Graph Used to Verify Decreasing Marginal Valuations**

$$= \sum_{i\in S} b_{i,k(i,S)} + \sum_{i\in T} b_{i,k(i,T)} - \sum_{i\in S\cup T} b_{i,k(i,S\cup T)} - \sum_{i\in S\cap T} b_{i,k(i,S\cap T)}$$
$$= b(S) + b(T) - b(S\cup T) - b(S\cap T).$$

Thus, if the shortest path in this graph has a nonnegative length, then the matrix bid has decreasing marginal valuations, and vice versa. Because the graph contains a number of nodes and arcs that are polynomial in the number of items ($O(m^4)$ and $O(m^5)$, respectively), it is clear that verifying whether a matrix bid b has decreasing marginal valuations can be done in polynomial time. □

If a bid function b satisfies the property that $b(S\cup T) + b(S\cap T) \geq b(S) + b(T)$ for all sets S and T, we call b supermodular, or, equivalently, b is said to have increasing marginal valuations. It is pointed out by de Vries and Vohra (2003) that if there are only two bid functions a bidder can have, both of them nondecreasing, integer valued, and supermodular, then the corresponding winner determination problem can be solved in polynomial time.

Notice that having b_{ik} values that are nondecreasing in the rows is no sufficient condition for a matrix bid to be supermodular. Choosing $S = \{x, z\}$ and $T = \{y, z\}$ in the matrix bid below illustrates this.

Item x	1		
Item y	1	2	
Item z	0	5	6

The same graph described in Theorem 3 can be used to verify whether a matrix bid is supermodular. Indeed, a longest path from source to sink with a nonpositive length implies that the matrix bid is supermodular. Because the graph is acyclic, its longest path can be found in polynomial time.

Corollary 2. *Verifying whether a matrix bid b has increasing marginal valuations can be done in polynomial time.*

3.4. Gross Substitutes Property

The gross substitutes property was introduced by Kelso and Crawford (1982) in the context of labor markets and applied to auctions by, e.g., Bevia et al. (1999) and Bikhchandani and Mamer (1997). The gross substitutes property makes use of a price vector p containing prices that are to be paid for each item i. Given a valuation function b, we can define the demand set $D(p)$ of the corresponding bidder given the current price vector p as

$$D(p) = \left\{ \arg\max_{S\subseteq G} \left(b(S) - \sum_{i\in S} p_i \right) \right\}. \qquad (6)$$

The gross substitutes property requires that a bidder will continue to demand items for which the price did not rise when other items have become more expensive. This condition can be defined more formally as follows.

Definition 5. A bid function b satisfies the gross substitutes property if for all price vectors $p \leq q$ (according to a pointwise comparison) and all sets $S \in D(p)$, there exists a set $T \in D(q)$ such that $\{i \in S: p_i = q_i\} \subseteq T$.

The gross substitutes property is stronger than submodularity, because Gul and Stacchetti (1999) show that each bid function that satisfies the gross substitutes condition is also submodular, whereas the converse is not true. Gul and Stacchetti (1999) also prove that in an auction where each bidder has a bid function that satisfies the gross substitutes property, there exists a price vector and an allocation such that every bidder receives a set of items that is in his demand set given these prices. This situation is known as a Walrasian equilibrium. Kelso and Crawford (1982) develop a fully polynomial approximation scheme for finding this Walrasian equilibrium (see also Nisan and Segal 2006).

Assuming that a bid function b has the gross substitutes property leads to a number of interesting results. Indeed, the LP-relaxation of a set-packing formulation for the winner determination problem of a combinatorial auction where all bidders have bid functions that satisfy the gross substitutes condition has an integral solution (Kelso and Crawford 1982, Bikhchandani et al. 2002). Furthermore, Murota and Tamura (2003) and Fujishige and Yang (2003) show that, given a valuation function that satisfies the gross substitutes property and a price vector, the bidder's demand set can be computed in polynomial time. Using the equivalence of separation and optimization (Grötschel et al. 1981), it follows that in this setting, the winner determination problem can be solved in polynomial time. Ausubel and Milgrom (2005) show that if every bidder has gross substitutes bid functions, the Vickrey-Clarke-Groves auction does not suffer from a number of weaknesses as, e.g., vulnerability to the use of multiple bidding identities by a single bidder (i.e., shill bidding).

The fact that the definition of the gross substitutes property is based on prices that should be paid for each item is somewhat awkward because a matrix bid specifies bids, and the price for a set of items is simply the winning bid for that set. However, Reijnierse et al. (2002) formulated the following equivalent characterization of the gross substitutes property, which is independent of prices. A bid function b satisfies the gross substitutes property if for all $S \subseteq G$ and $x, y, z \in G$ the following conditions hold:

$$b(S\cup\{x,y\}) - b(S\cup\{x\}) \le b(S\cup\{y\}) - b(S), \quad \text{and} \tag{7}$$

$$b(S\cup\{x,y\}) + b(S\cup\{z\}) \le \max(b(S\cup\{x,z\})+b(S\cup\{y\}), b(S\cup\{y,z\})+b(S\cup\{x\})). \tag{8}$$

Reijnierse et al. (2002) also show that it can be checked whether a bid function satisfies the gross substitutes property in a time complexity of $O(K\log^3(K))$, where K is the number of subsets of G, which equals 2^m. For matrix bids, however, this can be done in a time which is polynomial in the number of items, as witnessed by the following theorem.

Theorem 4. *Verifying whether a matrix bid b satisfies the gross substitutes property can be done in polynomial time.*

Proof. From the equivalence result by Reijnierse et al. (2002), it follows that a matrix bid b has the gross substitutes property if and only if conditions (7) and (8) are satisfied. Notice that condition (7) is in fact equivalent to submodularity. According to Theorem 3, checking whether a matrix bid b is submodular can be done in polynomial time. As for condition (8), it suffices to consider the setting where $x, y, z \notin S$. Indeed, the matrix bid auction is a single-unit auction, implying $b(S\cup\{x\}) = b(S)$ if $x \in S$. Using this, it is trivial to see that condition (8), where any subset of $\{x, y, z\}$ is in S, is satisfied for any submodular matrix bid. For the six possible relative rankings of items x, y, and z, we show that condition (8) must necessarily hold for four of the cases, and it can be verified by checking an associated shortest-path problem for the other two (nontrivial) cases. We will first assume that $r_x < r_y < r_z$ (i.e., item x is ranked higher than item y, which is ranked higher than item z), and afterwards we will discuss the other settings.

First, we show that for the setting with $r_x < r_y < r_z$, the following is true:

$$b(S\cup\{x,z\})+b(S\cup\{y\}) = b(S\cup\{y,z\})+b(S\cup\{x\}). \tag{9}$$

Indeed, all items i in S that are ranked higher than item x contribute $b(i, k(i, S))$ to each of the four terms in Equation (9), where $k(i, S)$ is the rank i has amongst the items in S. Item x adds $b(x, k(x, S\cup\{x,z\}))$ to the left-hand side of Equation (9), which equals $b(x, k(x, S\cup\{x\}))$ when added to the right-hand side. Items $i \in S$ that are ranked between x and y contribute $b(i, k(i, S)) + b(i, k(i, S)+1)$ to both sides of the equation. Items $i \in S$ ranked between y and z contribute $2b(i, k(i, S)+1)$ to both the left-hand and right-hand sides of Equation (9). Furthermore, item y also adds equal amounts to both sides of the equation—namely, $b(y, k(y, S\cup\{y\}))$ and $b(y, k(y, S\cup\{y,z\}))$. The same goes for item z, adding the equal terms $b(z, k(z, S\cup\{x,z\}))$ and $b(z, k(z, S\cup\{y,z\}))$ to the left-hand side and the right-hand side, respectively. Finally, items $i \in S$ ranked lower than z add $b(i, k(i, S)+2) + b(i, k(i, S)+1)$ to both sides of the equation. Using this result, condition (8) can be reformulated as

$$b(S\cup\{x,y\})+b(S\cup\{z\}) \le b(S\cup\{x,z\})+b(S\cup\{y\}). \tag{10}$$

Consider a graph containing a source, a sink, and nodes, indexed by (i, s, q). The index i refers to item i and ranges from 1 to m, whereas s ranges from 0 to r_i and q ranges from 0 to 3. There are arcs from each node (i, s, q) to $(i', s+1, q)$ and to $(i', s, q+1)$

Table 1 Cost on the Arcs from (i, s, q) to $(i', s+1, q)$ and to $(i', s, q+1)$, Depending on the Value for q

	$(i,s,q)\rightarrow(i',s+1,q)$	$(i,s,q)\rightarrow(i',s,q+1)$
$q=0$	$b_{i',s+1}+b_{i',s+1}-b_{i',s+1}-b_{i',s+1}=0$	$b_{i',s+1}-b_{i',s+1}=0$
$q=1$	$b_{i',s+2}+b_{i',s+1}-b_{i',s+2}-b_{i',s+1}=0$	$b_{i',s+1}-b_{i',s+2}$
$q=2$	$b_{i',s+2}+b_{i',s+2}-b_{i',s+3}-b_{i',s+1}$	$b_{i',s+2}-b_{i',s+1}$
$q=3$	$b_{i',s+3}+b_{i',s+2}-b_{i',s+3}-b_{i',s+2}=0$	(No such arc exists)

for all items i' ranked lower than i and insofar as $q+1\le 3$. Furthermore, there are arcs from the source to each node $(i,1,0)$ and $(i,0,1)$, and there are arcs from each node $(i,s,3)$ to the sink. Depending on the value for q, the arc from (i,s,q) to $(i',s+1,q)$ and the arc from (i,s,q) to $(i',s,q+1)$ have costs as presented in Table 1. The arcs from the source to each node $(i,1,0)$ and $(i,0,1)$, as well as all arcs to the sink, have a cost equal to zero.

The graph can be interpreted as follows. Each node (i,s,q) represents a state where s items ranked at least as high as item i are in set S. The index q keeps track of how many of the items x,y, and z are ranked at least as high as item i, and should be understood as follows:

$$q=0:\quad r_x>r_i,$$
$$q=1:\quad r_y>r_i>r_x,$$
$$q=2:\quad r_z>r_i>r_y,$$
$$q=3:\quad r_i>r_z.$$

Each path from source to sink determines which items are to be added to set S and which items are selected to play the roles of x, y, and z. If the path contains an arc from (i,s,q) or from the source to $(i',s+1,q)$, this indicates that item i' is added to the set S as the $(s+1)$th highest-ranked item. An arc from (i,s,q) or from the source to $(i',s,q+1)$ indicates that item i' is selected as item x, y, or z, for $q=0$, $q=1$, or $q=2$, respectively. The fact that there are only arcs from nodes $(i,s,3)$ to the sink guarantees that items x, y, and z are selected in each path from the source to the sink.

We now show that the length of a path from source to sink in this graph equals $b(S\cup\{x,z\})+b(S\cup\{y\})-b(S\cup\{x,y\})-b(S\cup\{z\})$. Each path from source to sink has exactly one arc that selects an item x (namely the arc where q increases from 0 to 1), one arc that selects an item y (q increases from 1 to 2), and one arc that selects an item z (q increases from 2 to 3). The other arcs in the path can be divided into four types: those arcs that add an item i ranked higher than x to the set S, those arcs that add an item i ranked between x and y to S, those arcs that add an item i ranked between y and z to S, and those arcs that add an item i ranked lower than z. From the costs in Table 1, it follows that the length of a path from source to sink equals

$$\begin{aligned}
&\sum_{i\in S:\, r_x>r_i}(b_{i,k(i,S\cup\{x,z\})}+b_{i,k(i,S\cup\{y\})}-b_{i,k(i,S\cup\{x,y\})}-b_{i,k(i,S\cup\{z\})})\\
&\quad+\sum_{i\in S:\, r_y>r_i>r_x}(b_{i,k(i,S\cup\{x,z\})}+b_{i,k(i,S\cup\{y\})}\\
&\qquad\qquad -b_{i,k(i,S\cup\{x,y\})}-b_{i,k(i,S\cup\{z\})})\\
&\quad+\sum_{i\in S:\, r_z>r_i>r_y}(b_{i,k(i,S\cup\{x,z\})}+b_{i,k(i,S\cup\{y\})}\\
&\qquad\qquad -b_{i,k(i,S\cup\{x,y\})}-b_{i,k(i,S\cup\{z\})})\\
&\quad+\sum_{i\in S:\, r_i>r_z}(b_{i,k(i,S\cup\{x,z\})}-b_{i,k(i,S\cup\{y\})}\\
&\qquad\qquad +b_{i,k(i,S\cup\{x,y\})}-b_{i,k(i,S\cup\{z\})})\\
&\quad+b_{x,k(x,S\cup\{x,z\})}-b_{x,k(x,S\cup\{x,y\})}-b_{y,k(y,S\cup\{x,y\})}\\
&\quad+b_{y,k(y,S\cup\{y\})}-b_{z,k(z,S\cup\{z\})}+b_{z,k(z,S\cup\{x,z\})}\\
&=\sum_{i\in S\cup\{x,z\}}b_{i,k(i,S\cup\{x,z\})}+\sum_{i\in S\cup\{y\}}b_{i,k(i,S\cup\{y\})}\\
&\quad-\sum_{i\in S\cup\{x,y\}}b_{i,k(i,S\cup\{x,y\})}-\sum_{i\in S\cup\{z\}}b_{i,k(i,S\cup\{z\})}\\
&=b(S\cup\{x,z\})+b(S\cup\{y\})-b(S\cup\{x,y\})-b(S\cup\{z\}).
\end{aligned}$$

Thus, if this graph has a shortest path with nonnegative length, then condition (8) is satisfied for every S and every $x,y,z\notin S$ such that $r_x<r_y<r_z$. Moreover, similar reasoning can be used to prove that this result is also valid if $r_y<r_x<r_z$.

We now show that for the setting with $r_x<r_z<r_y$, condition (8) is satisfied for any matrix bid, since

$$b(S\cup\{x,y\})+b(S\cup\{z\})=b(S\cup\{y,z\})+b(S\cup\{x\}).\quad (11)$$

Observe that all items i in S that are ranked higher than item x contribute $b(i,k(i,S))$ to each of the terms in Equation (11). Item x adds $b(x,k(x,S\cup\{x,y\}))$ to the left-hand side of Equation (11), which equals $b(x,k(x,S\cup\{x\}))$ when added to the right-hand side. Items $i\in S$ that are ranked between x and z contribute $b(i,k(i,S))+b(i,k(i,S)+1)$ to both sides of the equation. Also, item z adds an equal term to both sides of the equation—namely, $b(z,k(z,S\cup\{z\}))$ and $b(z,k(z,S\cup\{y,z\}))$ to the left-hand side and the right-hand side, respectively. Items $i\in S$ ranked between z and y contribute $2b(i,k(i,S)+1)$ to both the left-hand and right-hand sides of Equation (11). Furthermore, also item y adds equal amounts to both sides of the equation—namely, $b(y,k(y,S\cup\{x,y\}))$ and $b(y,k(y,S\cup\{y,z\}))$. Finally, items $i\in S$ ranked lower than y add $b(i,k(i,S)+2)+b(i,k(i,S)+1)$ to both sides of the equation. A similar reasoning can be used to show that equality (11) is also valid for a setting with $r_z<r_x<r_y$. Obviously, condition (8) is satisfied for any matrix bid that satisfies equality (11).

An analogous proof can be developed to show that condition (8) is also always satisfied for the settings with $r_y < r_z < r_x$ or $r_z < r_y < r_x$, since

$$b(S \cup \{x, y\}) + b(S \cup \{z\}) = b(S \cup \{x, z\}) + b(S \cup \{y\}). \quad (12)$$

We can conclude that it can be verified in polynomial time whether a matrix bid satisfies conditions (7) and (8) by solving a shortest-path problem on two graphs, with a number of nodes and arcs that are polynomial in the number of items ($O(m^4)$ and $O(m^5)$, and $O(m^2)$ and $O(m^3)$, respectively). If the shortest paths in both graphs have a nonnegative length, then the matrix bid has the gross substitutes property. Otherwise, if one of the shortest paths has a negative length, the matrix bid does not have the gross substitutes property. □

4. Computational Results

Theorems 1–4 allow us to develop a polynomial-time algorithm to verify whether a given matrix bid satisfies properties such as free disposal, complement freeness, decreasing marginal valuations (or submodularity), and the gross substitutes property. We considered five types of matrix bids, and for each type we looked at matrix bids with three, six, and nine items. We filled the entries of each matrix bid with randomly picked integer values as follows. For the random matrix bids, we drew the entries from a uniform distribution over the interval $[-20, 20]$. The positive matrix bid type consists of matrix bids with entries picked from a uniform distribution from 0 to 20. For the sparse matrix bids, there was a 50% probability of an entry with value 0, and a 50% probability of an entry with a value between 1 and 20. For the matrix bids with nonincreasing rows, we picked a value between 0 and 20 for the entries in the first column, and we picked values from a uniform distribution on an interval ranging from 0 to the entry on the column to the left for the other entries. The matrix bids with nonincreasing rows and columns were generated in a similar way, making sure of course that the entries of these matrix bids were nonincreasing in the rows as well as in the columns. For matrix bids of this type with six items, we imposed that the entries in the first three rows were between 20 and 40. For nine-item matrix bids with nonincreasing rows and columns, we picked values between 40 and 60 for the first three rows, and between 20 and 40 for the next three rows. This was done to avoid having the lowest rows, to a large extent, consist of 0s. For each of these settings, we verified for 500 generated matrix bids whether they satisfy the economic properties mentioned in §3. The algorithms were implemented using Visual C++, using ILOG CPLEX to solve shortest-path problems. The computational experiments were done on a Dell Latitude 1.66 GHz computer. All instances were solved in less than one second.

Table 2 Percentage of Matrix Bids with Three Items That Satisfy Economic Properties

	Free disp. (%)	Comp. free (%)	Submod. (%)	Gross subst. (%)
Random	1.0 ± 0.9	14.0 ± 3.0	7.0 ± 2.2	6.0 ± 2.1
Positive	50.0 ± 4.4	16.8 ± 3.3	8.6 ± 2.5	6.4 ± 2.1
Sparse	33.0 ± 4.1	26.0 ± 3.8	15.4 ± 3.2	13.8 ± 3.0
Nonincreasing rows	37.2 ± 4.2	100.0 ± 0.0	65.4 ± 4.2	51.0 ± 4.4
Nonincreasing rows and cols	100.0 ± 0.0	100.0 ± 0.0	84.2 ± 3.2	80.4 ± 3.5

Tables 2 and 3 show the percentage of the tested matrix bids that satisfy the various properties for settings with three and six items, respectively. Using a standard expression, we also derived a 95% confidence interval for each result. Even with three items, very few of our random matrix bids satisfy free disposal, which comes as no surprise, because a single negative entry already rules out free disposal. When negative entries are excluded, half of the three-item instances have free disposal, which also illustrates that this is not a sufficient condition (see §3.1). Sparse matrix bids have a higher chance of satisfying all properties, except free disposal. If we impose a more demanding structure such as nonincreasing rows, all matrix bids are complement free, as was argued in §3.2. This also explains the high number of matrix bids that satisfy submodularity or have the gross substitutes property, because complement freeness is a necessary condition for these properties. As argued in §3.2, the setting with nonincreasing rows and columns results in 100% free-disposal matrix bids. The extra requirement of having nonincreasing columns also results in a higher percentage of submodular matrix bids, and matrix bids having the gross substitutes property than compared with the nonincreasing rows type. Finally, notice as well that most properties occur with the three-item matrix bids; six-item matrix bids without much structure imposed rarely satisfy properties other than those for which sufficient conditions are satisfied. For the setting with

Table 3 Percentage of Matrix Bids with Six Items That Satisfy Economic Properties

	Free disp. (%)	Comp. free (%)	Submod. (%)	Gross subst. (%)
Random	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0
Positive	0.4 ± 0.6	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0
Sparse	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0
Nonincreasing rows	0.2 ± 0.4	100.0 ± 0.0	0.4 ± 0.6	0.0 ± 0.0
Nonincreasing rows and cols	100.0 ± 0.0	100.0 ± 0.0	8.2 ± 2.4	6.0 ± 2.1

nine items, all matrix bids lack the properties other than those satisfied as a result of a sufficient condition (i.e., free disposal for matrix bids with nonincreasing rows and columns and complement freeness for matrix bids with nonincreasing rows). Nevertheless, in practice, bids may possess more structure than randomly generated bids and be more likely to have the properties we discussed.

5. Conclusion

In this paper, we focused on the matrix bid auction, which is a combinatorial auction where a restriction is imposed on the preferences that can be expressed by a bidder. We argued that the matrix bid auction is interesting because it offers a compact way of representing preferences, and it allows us to capture structure, which results in faster computation. We also pointed out the relevance of recognizing whether a matrix bid has economic properties such as free disposal, complement freeness, decreasing marginal valuations, and the gross substitutes property, because it may allow efficient exact algorithms and good approximation results. In some cases, knowledge about these properties can also result in some insights on the eventual allocation. We revealed the relationship between verifying whether a matrix bid satisfies these properties and network problems such as the shortest-path problem. This allowed us to build a polynomial-time algorithm, which was tested on five types of randomly generated matrix bids, with three, six, and nine items. The results show that if little or no structure is imposed, matrix bids rarely satisfy any of these properties. Also, if the number of items increases, the chance of finding properties other than those for which the necessary conditions are fulfilled drastically decreases. Thus, matrix bids seem a priori not to be very restrictive, because most of them do not satisfy well-known economic properties. This should encourage auction designers to develop auctions that use matrix bids as bidding language, because it allows for efficient winner determination algorithms. However, given a particular application with its natural bidder preferences, an important question will be how well matrix bids are able to approximate the true preferences of the bidders.

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