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# Integer programming models for round robin tournaments

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ABSTRACT

Round robin tournaments are omnipresent in sport competitions and beyond. We investigate three integer programming formulations for scheduling a round robin tournament, one of which we call *the matching formulation*. We analytically compare their linear relaxations, and find that the relaxation of the matching formulation is stronger than the other relaxations, while still being solvable in polynomial time. In addition, we provide an exponentially sized class of valid inequalities for the matching formulation. Complementing our theoretical assessment of the strength of the different formulations, we also experimentally show that the matching formulation is superior on a broad set of instances. Finally, we describe a branch-and-price algorithm for finding round robin tournaments that is based on the matching formulation.



## 1. Introduction

Integer programming continues to be a very popular way to obtain a schedule for a round robin tournament. The ability to straightforwardly model such a tournament, and next solve the resulting formulation using an integer programming solver, greatly facilitates practitioners. Moreover, it is usually possible to add all kinds of specific local constraints to the formulation that help addressing particular challenges. We substantiate this claim of the widespread use of integer programming by mentioning some of the works that use integer programming to arrive at a schedule for a round robin tournament. Indeed, from the literature, it is clear that for national football leagues (which are predominantly organized according to a so-called double round robin format), integer programming-based techniques are used extensively to find schedules. Without claiming to be exhaustive we mention Alarcón et al. (2017), Della Croce & Oliveri (2006), Durán et al. (2021, 2007), Durán, Guajardo, & Sauré (2017), Goossens & Spieksma (2009), Rasmussen (2008), Recalde, Torres, & Vaca (2013), Ribeiro & Urrutia (2012). Other sport competitions that are organized in a round robin fashion (or a format close to a round robin) have also received ample attention: we mention Cocchi et al. (2018), Lambers, Rothuizen, & Spieksma (2021), and Raknes & Pettersen (2018) who use integer programming for scheduling volleyball leagues, Fleurent & Ferland (1993) who use integer programming for scheduling a hockey league, Kim (2019) and Bouzarth, Grannan, Harris, & Hutson (2022) for baseball leagues, Kostuk & Willoughby (2012) for Canadian football, Nemhauser & Trick (1998) and Westphal (2014) for basketball leagues. Further, there has been work on studying properties of the traditional formulation, among others, by Briskorn (2008), Trick (2003), and Briskorn & Drexl (2009a,b). Well-known surveys are given by Kendall, Knust, Ribeiro, & Urrutia (2010), Rasmussen & Trick (2008) and Goossens & Spieksma (2010); we also refer to Knust (2023), who maintains an elaborate classification of literature on sports scheduling. More recently, the international timetabling competition (Van Bulck & Goossens, 2022) featured a round robin sports timetabling problem, and most of the submissions for this competition used integer programming in some way to obtain a good schedule.

All this shows that integer programming is one of the most preferred ways to find schedules for competitions organized via a round robin format.

In this paper, we aim to take a fresh look at the problem of finding an optimal schedule for round robin tournaments using integer programming techniques. Depending upon how often a pair of teams is required to meet, different variations of a round robin tournament arise: in case each pair of teams meets once, the resulting format is called a Single Round Robin, in case each pair of teams is required to meet twice, we refer to the resulting variation as a Double Round Robin. These formats are the ones that occur most in practice; in general we speak of a $k$-Round Robin to describe the situation where each pair of teams is required to meet $k$ times.

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We have organized the paper as follows. In Section 2, we precisely define the problem corresponding to the Single Round Robin tournament, and we present three integer programming formulations for it. We call them the traditional formulation (Section 2.1), the matching formulation (Section 2.2), and the permutation formulation (Section 2.3), and we observe that their linear relaxations can be solved in polynomial time. We prove in Section 3 that the matching formulation is stronger than the other formulations. In Section 4 we provide a class of valid inequalities for the matching formulation. In Section 5, we generate instances of our problem with two goals in mind: (i) to experimentally assess the quality of the bounds found by our models (Section 5.2), and (ii) to report on the performance of a branch-and-price algorithm (Section 5.3). We conclude in Section 6.

## 2. Problem definition and formulations

In this section, we provide a formal definition of our problem and introduce the necessary terminology and notation. We focus on the so-called Single Round Robin (SRR) tournament, where every pair of teams has to meet exactly once. We refer to van Doornmalen, Hojny, Lambers, & Spieksma (2022) for a more general version of the problem, where every pair of teams has to meet $k$ times ($k \geq 1$).

Throughout the entire paper, we assume that $n$ is an even integer that denotes the number of teams; for reasons of convenience we assume $n \geq 4$. We denote the set of all teams by $T$. A *match* is a set consisting of two distinct teams and the set of all *matches* is denoted by $\mathcal{M}$, in formulae, $\mathcal{M} = \{m = \{i, j\} : i, j \in T,\ i \neq j\}$. We denote, for each $i \in T$, by $\mathcal{M}_i = \{\{i, j\} : j \in T \setminus \{i\}\}$ the set of matches played by team $i$. As we assume in this section that every pair of teams meets once, and as $n$ is even, the matches can be organized in $n-1$ rounds, which we denote by $R$; hence, we deal in this section with a *compact* single round robin tournament.

Prepared with this terminology and notation, we are able to provide a formal definition of the SRR problem.

**Problem 1** (SRR)**.** Given an even number $n \geq 4$ of teams with corresponding matches $\mathcal{M}$, a set of $n-1$ rounds $R$, as well as an integral cost $c_{m,r}$ for every match $m \in \mathcal{M}$ and round $r \in R$, the *single round robin (SRR)* problem is to find an assignment $\mathcal{A} \subseteq \mathcal{M} \times R$ of matches to rounds that minimizes the cost $\sum_{(m,r)\in\mathcal{A}} c_{m,r}$ such that every team plays a single match per round and each match is played in some round.

Since the SRR problem is NP-hard (see Briskorn, Drexl, & Spieksma, 2010, Van Bulck & Goossens, 2020, Easton, 2003), there does not exist a polynomial time algorithm to find an optimal assignment unless P = NP. For this reason, several researchers have investigated integer programming (IP) techniques for finding an optimal assignment of matches to rounds. We follow this line of research and discuss three different IP formulations for the SRR problem: a traditional formulation with polynomially many variables and constraints (Section 2.1) as well as two formulations that involve exponentially many variables (Sections 2.2 and 2.3). To the best of our knowledge, the latter models have not explicitly been discussed in the literature before; we point out though that Briskorn (2008) and Briskorn & Drexl (2009a) consider related formulations that include practical aspects such as breaks and home-away assignments.

### 2.1. The traditional formulation

The *traditional formulation* of the SRR problem has been discussed, among others, by Trick (2003) and Briskorn & Drexl (2009a,b). To model an assignment of matches to rounds, this formulation introduces, for every match $m \in \mathcal{M}$ and round $r \in R$, a binary decision variable $x_{m,r}$ to model whether match $m$ is played in round $r$ ($x_{m,r} = 1$) or not ($x_{m,r} = 0$). With these variables, problem SRR can be modeled as:

$$\min \sum_{m\in\mathcal{M}} \sum_{r\in R} c_{m,r} x_{m,r} \tag{T1}$$

$$\sum_{r\in R} x_{m,r} = 1, \qquad m \in \mathcal{M}, \tag{T2}$$

$$\sum_{m\in\mathcal{M}_i} x_{m,r} = 1, \qquad i \in T, r \in R, \tag{T3}$$

$$x_{m,r} \in \{0, 1\}, \qquad m \in \mathcal{M}, r \in R. \tag{T4}$$

Constraints (T2) ensure that each pair of teams meets once, and Constraints (T3) imply that each team plays in each round. This model has $O(n^2)$ constraints and $O(n^3)$ variables. Note that Constraints (T4) can be replaced by $x_{m,r} \in \mathbb{Z}_+$ as the upper bound $x_{m,r} \leq 1$ is implicitly imposed via Constraints (T2) and non-negativity of variables. The linear programming relaxation of (T) arises when we replace (T4) by $x_{m,r} \geq 0$; given an instance $I$ of SRR, we denote the resulting value by $\nu^{LP}_{tra}(I)$.

### 2.2. The matching formulation

A matching-based formulation is discussed by Briskorn & Drexl (2009a). Consider the complete graph that results when associating a node to each team, say $K_n = (T, \mathcal{M})$. Clearly, a single round of a feasible schedule can be seen as a perfect matching in this graph. This observation allows us to build a matching based formulation by introducing a binary variable for every perfect matching in $K_n$; we denote the set of all perfect matchings in $K_n$ by $\mathfrak{M}$. In fact, one can derive the following model also via a Dantzig–Wolfe decomposition Dantzig & Wolfe (1960), Desaulniers, Desrosiers, & Solomon (2005) applied to the traditional formulation (T). We will exploit this observation in Section 3.

We employ a binary variable $y_{M,r}$ for each perfect matching $M \in \mathfrak{M}$ and round $r \in R$. If $y_{M,r} = 1$, the model prescribes that matching $M$ is used for the schedule of round $r$, whereas $y_{M,r} = 0$ encodes that a different schedule is used. To be able to represent the cost of round $r \in R$, the total cost of all matches in $M$ is denoted by $d_{M,r} := \sum_{m\in M} c_{m,r}$, which leads to the model

$$\min \sum_{M\in\mathfrak{M}} \sum_{r\in R} d_{M,r} y_{M,r} \tag{M1}$$

$$\sum_{M\in\mathfrak{M}} y_{M,r} = 1, \qquad r \in R, \tag{M2}$$

$$\sum_{\substack{M\in\mathfrak{M}:\\ m\in M}} \sum_{r\in R} y_{M,r} = 1, \qquad m \in \mathcal{M}, \tag{M3}$$

$$y_{M,r} \in \{0, 1\}, \qquad M \in \mathfrak{M}, r \in R. \tag{M4}$$

Constraints (M2) ensure that a matching is selected in each round, while Constraints (M3) enforce that each pair of teams meets in some round. Similarly to the traditional formulation, we can replace (M4) by $y_{M,r} \in \mathbb{Z}_+$. In this way, the linear programming relaxation of (M) arises when replacing (M4) by $y_{M,r} \geq 0$; given an instance $I$ of SRR, the resulting value is denoted by $\nu^{LP}_{mat}(I)$. Notice that this formulation uses an exponential number of variables, as the number of matchings grows exponentially in $n$. Nevertheless, we will see below that computing $\nu^{LP}_{mat}$ is possible in polynomial time.

Formulation (M) is a variation of a model proposed by Briskorn & Drexl (2009a). Their model, however, also takes into account

whether or not a team plays home or away, and restricts the total number of times a team plays consecutive home or away matches. They observe that the pricing problem corresponding to their formulation boils down to finding a minimal cost perfect matching for each round. The proof also applies to formulation (M), implying the following result:

**Lemma 2.1** (cf. Briskorn & Drexl (2009a))**.** *The LP relaxation of the matching formulation (M) can be solved in polynomial time.*

### 2.3. The permutation formulation

Instead of fixing the schedule of a round, the *permutation formulation* fixes, for a given team, the order of the teams against which the given team plays its successive matches. That is, it introduces a variable for each team $i$ and each permutation of $T \setminus \{i\}$. We denote the set of all such permutations by $\Pi^{-i}$. Moreover, for a team $j \in T$ and round $r \in R$, denote the set of all permutations where $j$ occurs at position $r$ in the permutation by $\Pi^{-i}_{j,r}$. Permutations from $\Pi^{-i}_{j,r}$ thus encode that team $i$ plays against team $j$ in round $r$. For a permutation $\pi \in \Pi^{-i}$ and round $r \in R$, we refer to the opponent of team $i$ in round $r$ as $\pi_r \in T \setminus \{i\}$. The cost of a schedule encoded via permutations $\Pi^{-i}$ for a team $i \in T$ is then given by $e_{i,\pi} := \sum_{r\in R} c_{\{i,\pi_r\},r}$. Using binary variables $z_{i,\pi}$, where $i \in T$ and $\pi \in \Pi^{-i}$, that encode whether $i$ plays against its opponents in order $\pi$ ($z_{i,\pi} = 1$) or not ($z_{i,\pi} = 0$), the permutation formulation is

$$\min \frac{1}{2} \sum_{i\in T} \sum_{\pi\in\Pi^{-i}} e_{i,\pi} z_{i,\pi} \tag{P1}$$

$$\sum_{\pi\in\Pi^{-i}} z_{i,\pi} = 1, \qquad i \in T, \tag{P2}$$

$$\sum_{\pi\in\Pi^{-i}_{j,r}} z_{i,\pi} = \sum_{\pi\in\Pi^{-j}_{i,r}} z_{j,\pi}, \qquad \{i,j\} \in \mathcal{M}, r \in R, \tag{P3}$$

$$z_{i,\pi} \in \{0,1\}, \qquad i \in T, \pi \in \Pi^{-i}. \tag{P4}$$

Constraints (P2) ensure that a permutation is selected for each team, while Constraints (P3) enforce that, given a round and a pair of teams, these teams meet in that round, or they do not meet in that round. Due to rescaling the objective by $\frac{1}{2}$, we find the cost of an optimal SRR schedule. Moreover, we can again replace Constraint (P4) by $z_{i,\pi} \in \mathbb{Z}_+$. The linear programming relaxation of (P) then arises when replacing Constraints (P4) by $z_{i,\pi} \geq 0$; given an instance $I$ of SRR, we denote the resulting value by $\nu^{LP}_{per}(I)$.

Since this model has $n!$ variables, we investigate whether its LP relaxation can be solved efficiently.

**Lemma 2.2.** *The LP relaxation of the permutation formulation (P) can be solved in polynomial time.*

**Proof.** Due to the celebrated result by Grötschel, Lovász, & Schrijver (1981), it is sufficient to show that the separation problem for the constraints of the dual of the linear relaxation of Model (P) can be solved in polynomial time, see also Lübbecke (2011) for a discussion in the context of pricing problems.

We introduce dual variables $\alpha_i$ for each constraint of type (P2) and $\beta_{\{i,j\},r}$ for each constraint of type (P3). To normalize Constraint (P3), we assume it to be given by $\sum_{\pi\in\Pi^{-i}_{j,r}} z_{i,\pi} - \sum_{\pi\in\Pi^{-j}_{i,r}} z_{j,\pi} = 0$ with $i < j$. Then, the dual constraints are given by

$$\alpha_i + \sum_{\substack{r\in R:\\ i<\pi_r}} \beta_{\{i,\pi_r\},r} - \sum_{\substack{r\in R:\\ i>\pi_r}} \beta_{\{i,\pi_r\},r} \leq \frac{1}{2} e_{i,\pi} \qquad i \in T, \pi \in \Pi^{-i}.$$

If $i \in T$ is fixed, then, due to the definition of $e_{i,\pi}$, the separation problem for dual values $(\bar{\alpha}, \bar{\beta})$ is to decide whether there exists a permutation $\pi \in \Pi^{-i}$ such that

$$\sum_{\substack{r\in R:\\ i<\pi_r}} \bar{\beta}_{\{i,\pi_r\},r} - \sum_{\substack{r\in R:\\ i>\pi_r}} \bar{\beta}_{\{i,\pi_r\},r} - \frac{1}{2} \sum_{r\in R} c_{\{i,\pi_r\},r} > -\bar{\alpha}_i.$$

To answer this question, it is sufficient to find a permutation maximizing the left-hand side of this expression. Such a permutation can be found by computing a maximum weight perfect matching in the complete bipartite graph with node bipartition $(T \setminus \{i\}) \cup R$ and edge weights defined for each $j \in T \setminus \{i\}$ and $r \in R$ by

$$w_{j,r} = \begin{cases} -\frac{1}{2} c_{\{i,j\},r} + \bar{\beta}_{\{i,j\},r}, & \text{if } i < j, \\ -\frac{1}{2} c_{\{i,j\},r} - \bar{\beta}_{\{i,j\},r}, & \text{otherwise.} \end{cases}$$

Since this problem can be solved in polynomial time, the assertion follows by solving this problem for each of the $n$ teams. □

## 3. Comparing the strength of the different formulations

In the previous section, we have introduced three different models for finding an optimal schedule for problem SRR. While the traditional formulation contains both polynomially many variables and constraints, the matching and permutation formulation make use of an exponential number of variables. The aim of this section is to investigate whether the increase in the number of variables in comparison with the traditional formulation leads to a stronger formulation. We measure the strength of a formulation based on the value of its LP relaxation, where a higher value of the LP relaxation indicates a stronger formulation as the LP relaxation's value is closer to the optimum value of the integer program, as encapsulated by the following definitions.

**Definition 3.1.** Let $f$ and $g$ be mixed-integer programming formulations of the SRR problem and denote by $\nu^{\text{LP}}_f(I)$ and $\nu^{\text{LP}}_g(I)$ the value of the respective LP relaxations for an instance $I$ of SRR.

- We say that $f$ and $g$ are *relaxation-equivalent* if, for each instance $I$ of problem SRR, the value of the linear programming relaxations are equal, i.e., $\nu^{\text{LP}}_f(I) = \nu^{\text{LP}}_g(I)$.
- We say that $f$ is stronger than (or dominates) $g$ if (i) for each instance $I$ of problem SRR, $\nu^{\text{LP}}_f(I) \geq \nu^{\text{LP}}_g(I)$, and (ii) there exists an instance $I$ of problem SRR for which $\nu^{\text{LP}}_f(I) > \nu^{\text{LP}}_g(I)$.

We now proceed by formally comparing the strength of the formulations from Section 2 using the terminology of these definitions. We state our results using three lemmata, and summarize all our results in Theorem 3.5.

First, we show that the traditional and permutation formulation have equivalent LP relaxations.

**Lemma 3.2.** *The permutation formulation (P) is relaxation-equivalent to the traditional formulation (T).*

**Proof.** To prove this lemma, we show that there is a one-to-one correspondence between feasible solutions of the traditional formulation's and the permutation formulation's LP relaxations that preserves the objective value. First, we construct a solution of the LP relaxation of the traditional formulation from a solution $z$ of the LP relaxation of the permutation formulation. To this end, define for each $\{i,j\} \in \mathcal{M}$ and $r \in R$ a solution $x \in \mathbb{R}^{\mathcal{M}\times R}$ via $x_{\{i,j\},r} = \sum_{\pi\in\Pi^{-i}_{j,r}} z_{i,\pi}$. Note that $x$ is non-negative as all $z$-variables are non-negative. Moreover, it is well-defined as $\sum_{\pi\in\Pi^{-i}_{j,r}} z_{i,\pi} = \sum_{\pi\in\Pi^{-j}_{i,r}} z_{j,\pi}$ due to (P3). Finally, all constraints of type (T2) and (T3) are satisfied since

$$\text{for each } m \in \mathcal{M}: \quad \sum_{r\in R} x_{\{i,j\},r} = \sum_{r\in R}\sum_{\pi\in\Pi^i_{j,r}} z_{i,\pi} = \sum_{\pi\in\bigcup_{r\in R}\Pi^i_{j,r}} z_{i,\pi} = \sum_{\pi\in\Pi^{-i}} z_{i,\pi} \overset{P2}{=} 1,$$

$$\text{for each } i\in T,\ r\in R: \quad \sum_{j\in T\setminus\{i\}} x_{\{i,j\},r} = \sum_{j\in T\setminus\{i\}}\sum_{\pi\in\Pi^i_{j,r}} z_{i,\pi} = \sum_{\pi\in\Pi^{-i}} z_{i,\pi} \overset{P2}{=} 1.$$

We conclude the proof by constructing a feasible solution for the LP relaxation of the permutation formulation from a feasible solution $x$ of the traditional formulation's LP relaxation.

Let $x$ be such a solution and let $i\in T$. Consider the matrix $X^i\in\mathbb{R}^{(T\setminus\{i\})\times R}$ with entries $X^i_{j,r}=x_{\{i,j\},r}$. Due to all constraints of the traditional formulation's LP relaxation, $X^i$ is a doubly stochastic matrix and is thus contained in the Birkhoff polytope, see Ziegler (1995). Consequently, $X^i$ can be written as a convex combination of all permutation matrices. That is, if $P^{i,\pi}$ is the permutation matrix associated with $\pi\in\Pi^{-i}$, there exist multipliers $\lambda^i_\pi\geq 0$, $\pi\in\Pi^{-i}$, such that $X^i=\sum_{\pi\in\Pi^{-i}}\lambda^i_\pi P^{i,\pi}$ and $\sum_{\pi\in\Pi^{-i}}\lambda^i_\pi=1$. Based on these multipliers, we define a solution $z$ of the permutation formulation via $z_{i,\pi}=\lambda^i_\pi$. To conclude the proof, we need to show that this solution $z$ is feasible for the permutation formulation's LP relaxation and has the same objective value as $x$. Observe that $z$ is non-negative since all $\lambda$'s are non-negative. Constraints (P2) and (P3) are satisfied as

$$\text{for each } i\in T: \quad \sum_{\pi\in\Pi^{-i}} z_{i,\pi} = \sum_{\pi\in\Pi^{-i}}\lambda^i_\pi = 1,$$

$$\text{for each } \{i,j\}\in\mathcal{M},\ r\in R: \quad \sum_{\pi\in\Pi^i_{j,r}} z_{i,\pi} = \sum_{\pi\in\Pi^i_{j,r}}\lambda^i_\pi = x_{\{i,j\},r} = \sum_{\pi\in\Pi^j_{i,r}}\lambda^j_\pi = \sum_{\pi\in\Pi^{-j}} z_{j,\pi}$$

since $\sum_{\pi\in\Pi^{-i}}\lambda^i_\pi=1$ and $x_{\{i,j\},r}$ is a convex combination of permutation matrices that assign team $j$ (or $i$) to round $r$, respectively. Consequently, $z$ is feasible for the permutation formulation's LP relaxation. Finally, both $x$ and $z$ have the same objective value because

$$\begin{aligned}\frac{1}{2}\sum_{i\in T}\sum_{\pi\in\Pi^{-i}} e_{i,\pi}z_{i,\pi} &= \frac{1}{2}\sum_{i\in T}\sum_{\pi\in\Pi^{-i}} e_{i,\pi}\lambda^i_\pi = \frac{1}{2}\sum_{i\in T}\sum_{\pi\in\Pi^{-i}}\sum_{r\in R} c_{\{i,\pi_r\},r}\lambda^i_\pi\\ &= \frac{1}{2}\sum_{i\in T}\sum_{j\in T\setminus\{i\}}\sum_{r\in R} c_{\{i,j\},r}\sum_{\substack{\pi\in\Pi^{-i}:\\ \pi_r=j}}\lambda^i_\pi P^{i,\pi}_{\pi_r,r}\\ &= \frac{1}{2}\sum_{i\in T}\sum_{j\in T\setminus\{i\}}\sum_{r\in R} c_{\{i,j\},r}x_{\{i,j\},r}\\ &= \sum_{\{i,j\}\in\mathcal{M}}\sum_{r\in R} c_{\{i,j\},r}x_{\{i,j\},r}.\end{aligned}$$

This proves that both formulations are relaxation-equivalent. □

Next, we turn our focus to the matching formulation and compare it with the traditional formulation (and thus, by the previous lemma, also with the permutation formulation).

**Lemma 3.3.** *For each $n\geq 6$, the matching formulation (M) is stronger than the traditional formulation (T).*

**Proof.** We decompose matrix $x\in[0,1]^{\mathcal{M}\times R}$ into the column vectors $x^1,\dots,x^{|R|}$ representing the matches of a specific round. Observe that, for fixed round index $r\in R$, the integer points in

$$P^r=\left\{x^r\in[0,1]^{\mathcal{M}}:\sum_{m\in\mathcal{M}_i}x^r_m=1 \text{ for all } i\in T\right\} \tag{4}$$

correspond to the perfect matchings in $K_n$. From (4), we can thus derive a formulation equivalent to (T):

$$\left\{x=(x^1,\dots,x^{|R|})\in[0,1]^{\mathcal{M}\times R}: x \text{ satisfies (T2) and } x^r\in P^r \text{ for all } r\in R\right\}. \tag{5}$$

Let $P^r_I$ denote the integer hull of $P^r$, i.e., $P^r_I$ is the perfect matching polytope. Since solutions are binary and $P^r_I\subseteq P^r$, we can strengthen the LP relaxation of (T) by replacing $x^r\in P^r$ by $x^r\in P^r_I$ in (5). Applying Dantzig–Wolfe decomposition to $x^r\in P^r_I$ results in expressing every partial solution $x^r$ as a convex combination of incidence vectors of perfect matchings. The latter is exactly the matching formulation (2), which by the previous arguments cannot be weaker than the traditional formulation (T).

To prove that the matching formulation dominates the traditional formulation for any even integer $n\geq 6$, we distinguish three cases. In the first case, assume $n\geq 10$. Consider the pairs of teams given by

$$E=\{\{1,2\},\{2,3\},\{1,3\}\}\cup\{\{4,5\},\{5,6\},\{4,6\}\}\cup \{\{7,8\},\{8,9\},\dots,\{n-1,n\},\{7,n\}\}.$$

Interpreting $E$ as the edges of an undirected graph, $E$ defines three connected components consisting of two 3-cycles and an even cycle. We construct an instance of the SRR problem by specifying the cost function $c\in\mathbb{R}^{\mathcal{M}\times R}$ via

$$c_{m,r}=\begin{cases}1, & \text{if } m\notin E \text{ and } r\in\{1,2\},\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to verify that $x\in\mathbb{R}^{\mathcal{M}\times R}$ given by

$$x_{m,r}=\begin{cases}\frac{1}{2}, & \text{if } m\in E \text{ and } r\in\{1,2\},\\ 0, & \text{if } m\notin E \text{ and } r\in\{1,2\},\\ \frac{1}{n-3}, & \text{otherwise,}\end{cases}$$

is feasible for the LP relaxation of the traditional formulation and has objective value 0. Hence, $x$ is optimal.

Solving the LP relaxation of the matching formulation for this instance, however, results in an objective value that is at least 2. Indeed, each perfect matching $M\in\mathfrak{M}$ contains at least one match $m\in M$ with $m\in\{\{i,j\}:(i,j)\in\{1,2,3\}\times\{4,5,\dots,n\}\}$. Since $c_{m,1}=c_{m,2}=1$ for such a match, it follows that in both rounds 1 and 2, matchings are selected with total weight at least 1 due to (M3), leading to a solution with total cost at least 2.

In the second case, we consider $n=6$. To prove the statement, we use the same construction as before, however, we do not require the even cycle anymore. That is, $E$ defines two 3-cycles and the argumentation remains the same as before.

In the last case $n=8$, we consider the set of pairs

$$E=\left\{\{1,2\},\{1,3\},\{2,3\}\right\}\cup\left\{\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{4,8\}\right\}.$$

If we interpret $E$ as edges of an undirected graph, the corresponding graph has two connected components being a 3-cycle and a 5-cycle, respectively. We choose the cost-coefficients $c\in\mathbb{R}^{\mathcal{M}\times R}$ to be

$$c_{m,r}=\begin{cases}1, & \text{if } m\notin E \text{ and } r\in\{1,2\},\\ 0, & \text{otherwise.}\end{cases}$$

Simple calculations show that an optimal solution of the traditional formulation's LP relaxation is $x\in\mathbb{R}^{\mathcal{M}\times R}$ with

$$x_{m,r}=\begin{cases}\frac{1}{2}, & \text{if } m\in E \text{ and } r\in\{1,2\},\\ 0, & \text{if } m\notin E \text{ and } r\in\{1,2\},\\ \frac{1}{5}, & \text{otherwise,}\end{cases}$$

which has objective value 0, whereas the matching formulation's LP relaxation has value 2. □

The previous two lemmata completely characterize the relative strength of the three different formulations except for $n = 4$. The status of this case is settled in the next lemma.

**Lemma 3.4.** *For $n = 4$, the traditional formulation and the matching formulation are relaxation-equivalent.*

**Proof.** Using the same notation as in Lemma 3.3, note that $P^r = P^r_I$ if $n = 4$. Hence, Dantzig-Wolfe reformulation used to derive the matching formulation from the traditional formulation does not strengthen the latter. Both are thus relaxation-equivalent. □

Summarizing the previous results of this section, we can provide a complete comparison of the strength of the traditional, matching, and permutation formulation.

**Theorem 3.5.** *For each $n \geq 6$, the traditional and permutation formulation are relaxation-equivalent, whereas the matching formulation is stronger than either of them. For $n = 4$, the traditional, matching, and permutation formulation for problem SRR are relaxation-equivalent.*

Besides verifying that all three models are equivalent for $n = 4$, we can also show that the matching formulation's integer hull is already completely characterized by (M2), (M3), as well as non-negativity inequalities for all variables.

**Proposition 3.6.** *For $n = 4$, Eqs. (M2) and (M3) as well as non-negativity inequalities define an integral polyhedron. That is, the matching formulation's LP relaxation coincides with its integer hull.*

**Proof.** To prove the proposition's statement, we show that the constraint matrix of (M) is totally unimodular. The result follows then by the Hoffman–Kruskal theorem Schrijver (1987) as all right-hand side values in (M) are integral.

For $n = 4$, the set of all matchings $\mathfrak{M}$ consists of exactly the three matchings

$$M_1 = \{\{1,2\},\{3,4\}\},\ M_2 = \{\{1,3\},\{2,4\}\},\ M_3 = \{\{1,4\},\{2,3\}\}.$$

The non-trivial constraints from Formulation (M) are Eqs. (M2) and (M3), which yield system

$$\begin{pmatrix}
1 & & & 1 & & & 1 & & \\
 & 1 & & & 1 & & & 1 & \\
 & & 1 & & & 1 & & & 1 \\
1 & 1 & 1 & & & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & & & \\
1 & 1 & 1 & & & & & &
\end{pmatrix}
\begin{pmatrix}
y_{M_1,1} \\ y_{M_1,2} \\ y_{M_1,3} \\ y_{M_2,1} \\ y_{M_2,2} \\ y_{M_2,3} \\ y_{M_3,1} \\ y_{M_3,2} \\ y_{M_3,3}
\end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{array}{l} ((\text{M2}), r = 1) \\ ((\text{M2}), r = 2) \\ ((\text{M2}), r = 3) \\ ((\text{M3}), m = \{1,2\}) \\ ((\text{M3}), m = \{1,3\}) \\ ((\text{M3}), m = \{1,4\}) \\ ((\text{M3}), m = \{2,3\}) \\ ((\text{M3}), m = \{2,4\}) \\ ((\text{M3}), m = \{3,4\}) \end{array}$$

Note that the last three equations are redundant and can be removed. The constraint matrix of the remaining equations is the node-edge incidence matrix of a bipartite graph and hence totally unimodular, which concludes the proof. □

Thus, for $n = 4$, simply solving the LP-relaxation of the matching formulation by the simplex method, suffices to find an optimum integral solution.

## 4. Strengthening the formulations

In this section, we continue our investigations of the structure of the formulations. In Section 4.1, we derive an exponentially sized class of valid inequalities for the matching formulation. Also, we show in Section 4.2 that adding the so-called odd-cut inequalities to the traditional formulation yields a formulation that is actually relaxation-equivalent to the matching formulation. The latter is an idea that has also been described by Trick (2003).

### 4.1. Strengthening the matching formulation

Observe that Theorem 3.5 does not rule out the possibility that, for $n \geq 6$, every vertex of the LP relaxation of the matching formulation is integral. That, however, is not the case. In Example 4.1, we give an example of a fractional solution of the LP relaxation of the matching formulation that corresponds to an extreme point of its corresponding polytope.

**Example 4.1.** Let $n = 6$. Then, the set of teams and rounds is given by $T = \{1,\dots,6\}$ and rounds $R = \{1,\dots,5\}$, respectively. In Fig. 1, we depict a fractional solution of the matching formulation's LP relaxation. For each round $r \in R$, we provide two perfect matchings between the teams $T$, the blue and green (dashed) matching $M$, whose corresponding variables $y_{M,r}$ have value $\frac{1}{2}$ in the corresponding solution; all remaining variables have value 0. It is easy to verify that this fractional solution $y^\star$ is indeed feasible for the LP relaxation of (M). In fact, we claim that it is an extreme point.

We now proceed by exhibiting a class of inequalities valid for the matching formulation. To describe our class of valid inequalities, consider the following lemma.

**Lemma 4.2.** *Let $m_1, m_2 \in \mathcal{M}$ be disjoint and let $r' \in R$. Then,*

$$\sum_{r \in R \setminus \{r'\}} \sum_{\substack{M \in \mathfrak{M}: \\ m_1 \in M \text{ or } m_2 \in M}} y_{M,r} + \sum_{\substack{M \in \mathfrak{M}: \\ m_1 \notin M \text{ or } m_2 \notin M}} y_{M,r'} + \sum_{\substack{M \in \mathfrak{M}: \\ m_1, m_2 \in M}} 2y_{M,r'} \geq 2 \quad (6)$$

*is a valid inequality for (M). In particular, it is a Chvátal–Gomory cut derived from the LP relaxation of (M).*

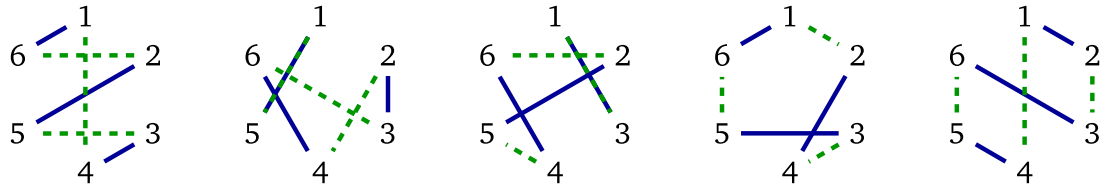
**Proof.** It is sufficient to prove that (6) is indeed a Chvátal–Gomory cut. To this end, we multiply Eq. (M2) for round $r'$ and Eq. (M3) for matches $m_1$ and $m_2$ by $\frac{1}{2}$ and sum the resulting equations to obtain

$$\sum_{r \in R \setminus \{r'\}} \sum_{\substack{M \in \mathfrak{M}: \\ m_1 \in M \text{ or } m_2 \in M}} \frac{C_{M,r}}{2} y_{M,r} + \sum_{\substack{M \in \mathfrak{M}: \\ m_1 \notin M \text{ or } m_2 \notin M}} \frac{1 + C_{M,r}}{2} y_{M,r'} + \sum_{\substack{M \in \mathfrak{M}: \\ m_1, m_2 \in M}} \frac{3}{2} y_{M,r'} = \frac{3}{2},$$

where $C_{M,r} = |M \cap \{m_1, m_2\}|$. Since all $y$-variables are non-negative, we can turn this equation into a $\geq$-inequality by rounding up the left-hand side coefficients. Moreover, since in a feasible solution for (M) all variables attain integer values, we can increase the right-hand side from $\frac{3}{2}$ to 2, which yields the desired inequality. □

**Example 4.3** (Example 4.1 continued)**.** Using this class of inequalities, we can show that the point $y^\star$ presented before is indeed not contained in the matching formulation's integer hull. Select $r' = 1$, $m_1 = \{1,6\}$, and $m_2 = \{3,5\}$. To see that $y^\star$ violates the corresponding inequality, let $M$ be the blue and $M'$ be the green matching of the first round as well as $M''$ the blue matching of round 4. Then, the corresponding inequality's left-hand side evaluates in $y^\star$ to $y^\star_{M'',4} + y^\star_{M,1} + y^\star_{M',1} = \frac{3}{2}$. Hence, $y^\star$ violates the corresponding inequality as $\frac{3}{2} \not\geq 2$.

Note that Inequality (6) is a so-called $\{0, \frac{1}{2}\}$-cut (Caprara & Fischetti, 1996) as all multipliers used in the derivation are $\frac{1}{2}$ (and 0 for inequalities/equations that have not been used). By taking more




**(a)** round 1 **(b)** round 2 **(c)** round 3 **(d)** round 4 **(e)** round 5

**Fig. 1.** A feasible point for the LP relaxation of Formulation (M).

equations in the generation of a valid inequality into account, we can generalize (6) to an exponentially large class of inequalities as follows.

**Proposition 4.4.** *Let $A \subseteq \mathcal{M}$ be a set of pairwise disjoint matches and let $B \subseteq R$. If $|A| + |B|$ is odd, then*

$$\sum_{M\in\mathfrak{M}} \sum_{r\in B} \left\lceil \frac{1+|M\cap A|}{2} \right\rceil y_{M,r} + \sum_{M\in\mathfrak{M}} \sum_{r\in R\setminus B} \left\lceil \frac{|M\cap A|}{2} \right\rceil y_{M,r} \geq \frac{1+|A|+|B|}{2}, \tag{7}$$

*is a valid inequality for (M). In particular, it is a Chvátal–Gomory cut derived from the LP relaxation of (M).*

**Proof.** We follow the line of the proof of Lemma 4.2 and multiply each constraint of type (M2) with index in $A$ and each constraint of type (M3) with index in $B$ by $\frac{1}{2}$ and sum all resulting equations. This leads to

$$\sum_{j=1}^{n/2} \sum_{\substack{M\in\mathfrak{M}:\\ |M\cap A|=j}} \sum_{r\in B} \frac{1+j}{2} y_{M,r} + \sum_{j=1}^{n/2} \sum_{\substack{M\in\mathfrak{M}:\\ |M\cap A|=j}} \sum_{r\in R\setminus B} \frac{j}{2} y_{M,r} = \frac{|A|+|B|}{2}.$$

Since all $y$-variables are non-negative, we derive the inequality

$$\sum_{M\in\mathfrak{M}} \sum_{r\in B} \left\lceil \frac{1+|M\cap A|}{2} \right\rceil y_{M,r} + \sum_{M\in\mathfrak{M}} \sum_{r\in R\setminus B} \left\lceil \frac{|M\cap A|}{2} \right\rceil y_{M,r} \geq \frac{|A|+|B|}{2},$$

and by integrality of the $y$-variables, we can round up the right-hand side, which leads to the desired inequality. □

While Inequalities (6) can trivially be separated in polynomial time, an efficient separation algorithm for (7) is not immediate. We leave the complexity status of separating (7) open for future research.

### 4.2. Strengthening the traditional formulation

In this section, we consider the impact of adding so-called odd-cut inequalities (an idea described by Trick, 2003) to the traditional formulation. While Trick (2003) used these inequalities to strengthen the traditional formulation, we also provide theoretical implications.

Recall the definition of $P^r$, $r \in R$, from the proof of Lemma 3.3. A classical observation from matching theory is that there exist extreme points of $P^r$ such that each edge (match) of an odd cycle in $K_n$ has a weight of $\frac{1}{2}$. That is, the traditional formulation can assign an odd cycle of length $k$ a weight of $\frac{k}{2}$. Such a solution, however, cannot be written as a convex combination of integer feasible solutions, because each such solution defines a perfect matching on the matches of a fixed round, i.e., the total weight of an odd cycle can be at most $\frac{k-1}{2}$. To strengthen the traditional formulation, one can thus add facet defining inequalities for the perfect matching polytope $P_M$ to Model (T), which results in the additional inequalities

$$\sum_{i\in U} \sum_{j\in T\setminus U} x_{\{i,j\},r} \geq 1, \qquad U \subseteq T \text{ with } |U| \text{ odd}, r \in R, \tag{8}$$

which correspond to the odd-cut inequalities for the matching polytope and can be separated in polynomial time. In particular, due to the classical result by Edmonds (1965), adding Inequalities (8) to $P^r$ yields $P_I^r$.

**Lemma 4.5.** *Let $n \geq 6$. The traditional formulation (T) extended by (8) is relaxation-equivalent to the matching formulation.*

**Proof.** Using the notation from the proof Lemma 3.3, adding (8) to (M) allows us to replace $x^r \in P^r$ in (5) by $x^r \in P_I^r$. Consequently, as the matching formulation is derived from $x^r \in P_I^r$ via Dantzig–Wolfe decomposition, both formulations are relaxation-equivalent. □

**Remark 4.6.** Since the traditional and permutation formulation are equivalent, one might wonder whether also the permutation formulation can be enhanced by odd-cut inequalities. Indeed, using the transformation $x_{\{i,j\},r} = \sum_{\pi\in\Pi_{j,r}^i} z_{i,\pi}$ as in the proof of Lemma 3.2, one can show that the corresponding version of odd-cut inequalities is given by

$$\sum_{i\in U} \sum_{j\in T\setminus U} \sum_{\pi\in\Pi_{j,r}^i} z_{i,\pi} \geq 1, \qquad U \subseteq T \text{ with } |U| \text{ odd}, r \in R,$$

and that the enhanced traditional and permutation formulation are equivalent.

## 5. Computational results

In this section, we report the outcomes of our computational experiments. Section 5.1 describes the test set that we have used in our experiments. Afterwards, we investigate the quality of the LP relaxations of the different models and compare their corresponding values in Section 5.2. Finally, we discuss our experience with solving instances of the SRR problem using the matching formulation, i.e., not only solving the LP relaxation but also the corresponding integer program. To this end, we have implemented a branch-and-price algorithm whose details are described in Section 5.3. Note that Briskorn & Drexl (2009a) also report computational experience with a branch-and-price algorithm. While they report on a model that also takes breaks and home-away assignments into account, we focus on the plain models for SRR.

### 5.1. Test set

We have generated 1000 instances of the SRR problem to evaluate the quality of the LP relaxations of the different models. Both our instances and implementation are publicly available at https://github.com/JasperNL/round-robin; all experiments have been conducted using the code with githash `1657b4d7`. Our test

set comprises of instances of different sizes, which are parameterized by a tuple $(n, \rho)$ and have cost coefficients attaining values 0 or 1. Parameter $n$ encodes the number of teams and has range $n \in \{6, 12, 18, 24\}$; parameter $\rho$ controls the number of 1-entries in the objective. More precisely, we pick a set of match-round pairs $\mathcal{M} \times R$ of size $\lfloor \rho \cdot |\mathcal{M} \times R| \rfloor$ uniformly at random, denoted by $S \subseteq \mathcal{M} \times R$, where $\rho \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$. The generated instance consists of $n$ teams and has cost coefficients $c_{m,r} = 1$ if $(m, r) \in S$ and $c_{m,r} = 0$ otherwise. For each combination of $n \in \{6, 12, 18, 24\}$ and $\rho \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ we have generated 50 instances.

### 5.2. A computational comparison of the linear relaxations

In this section, we provide a computational comparison between the LP relaxation values of the traditional formulation (T) (and thus by Lemma 3.2 also of the permutation formulation) and LP relaxation values of the matching formulation (M), and compare these to the actual optimal (or best found) integral solutions. Before we discuss our numerical results, we provide details about our implementation as well as on how we find optimal integral solutions.

*Implementation details.* To find the LP relaxation values, we implemented both formulations in Python 3 using the `PySCIPOpt` 4.1.0 package (Maher et al., 2016) for `SCIP` 8.0.0 (Bestuzheva et al., 2021), with `CPLEX` 20.1.0.0 as LP solver. The traditional formulation is implemented as a compact model. For the matching formulation, we use a column generation procedure that receives a subset of all variables, solves the corresponding LP relaxation restricted to these variables, and adds further variables until it can prove that an optimal LP solution has been found. To identify whether new variables need to be added, we solve the so-called pricing problem, which corresponds to separating a corresponding solution of the dual problem.

As a result of Lemma 2.1, we can efficiently solve the pricing problem. Let $r \in R$ and $m \in \mathcal{M}$, and let $\alpha_r$ and $\beta_m$ denote the dual variable values associated with Constraints (M2) and (M3), respectively. For each round $r \in R$, consider the complete graph with vertex set $T$ and the weight of edge $m \in \mathcal{M}$ as $\beta_m - c_{m,r}$. For $M \in \mathfrak{M}$, variable $y_{M,r}$ has negative reduced cost if and only if the total weight of this perfect matching is larger than $-\alpha_r$. In our implementation, for each round we compute a maximal weight perfect matching, and add the associated variable if it has negative reduced cost.

We start with the empty set of variables, which means that the primal problem is initially infeasible. We resolve infeasibility by adding variables to the problem that are associated with a dual constraint that violate a dual unbounded ray of this infeasible problem. The column generation procedure has been embedded in a so-called `pricer` plug-in of `SCIP`, which adds newly generated variables to the matching formulation. The maximal weight perfect matchings are computed using `NetworkX` 2.5.1, which provides an implementation of Edmonds' blossom algorithm.

*Finding optimal integer solutions* To obtain the optimal integer solution value of as many instances as possible, we have used two different solvers to solve the integer program of Model (T). On the one hand, we have used `SCIP` as described in the above setup. On the other hand, we have modeled (T) using `Gurobi` 9.1.2 via its Python 3 interface. For each instance and solver, we have imposed a time limit of 48 hours to find an optimal integer solution. Using `SCIP`, we managed to solve 852 of the 1000 instances to optimality. With `Gurobi`, we were able to solve 866 of the 1000 instances to optimality. Out of the instances that were not solved to optimality by all solvers, there were 45 instances where `SCIP` found a better primal objective value, and 79 instances where `Gurobi` found a better primal objective value. All experiments have been run on a compute cluster with identical machines, using one (resp. two) thread(s) on Xeon Platinum 8260 processors, with 10.7 gigabyte (resp. 21.4 gigabyte) memory, respectively for `SCIP` and `Gurobi`.

*Numerical results* Table 1 shows the aggregated computational results of our experiments. For each number of teams $n$ and ratio $\rho$, we provide the average of the objective values of the relaxation of the traditional formulation (column "$v^{\mathrm{LP}}_{\mathrm{tra}}$"), the average of the objective values of the relaxation of the matching formulation (column "$v^{\mathrm{LP}}_{\mathrm{mat}}$"), and the average optimum value (column "$v^{\mathrm{IP}}$"). Notice that for $n = 24$ we have not been able to solve all instances to optimality; in this case, we use the value of the best known solution instead of the (unknown) optimum for that instance in the $v^{\mathrm{IP}}$ column. Recall that each value is an average over 50 instances. The number of optimally solved instances (resp. instances not terminating within the time limit) are shown in column "O" (resp. "T").

To be able to assess the strength of the matching formulation compared to the traditional formulation, we focus, in the right side of the table, on those instances for which $v^{\mathrm{LP}}_{\mathrm{tra}} < v^{\mathrm{IP}}$; their number (out of 50) is given in the column labeled "#". From this column, we see that the fraction $\rho$ that leads to instances with a gap between $v^{\mathrm{LP}}_{\mathrm{tra}}$ and $v^{\mathrm{IP}}$ slowly increases with $n$. Indeed, for $n = 6$, most instances do not have a gap, for $n = 12$, almost all instances with $\rho \in \{0.6, 0.7, 0.8\}$ have a gap, and for $n = 18$, almost all instances with $\rho \in \{0.7, 0.8, 0.9\}$ have a gap.

We use the notion of the *relative gap* that is closed by the matching formulation relative to the traditional formulation, given by

$$\mathrm{rgap}(I) := \frac{v^{\mathrm{LP}}_{\mathrm{mat}}(I) - v^{\mathrm{LP}}_{\mathrm{tra}}(I)}{v^{\mathrm{IP}}(I) - v^{\mathrm{LP}}_{\mathrm{tra}}(I)} \text{ for an instance } I \text{ of SRR with } v^{\mathrm{IP}}(I) - v^{\mathrm{LP}}_{\mathrm{tra}}(I) > 0.$$

A value of zero for $\mathrm{rgap}(I)$ implies that the relaxation values of the traditional formulation and the matching formulation are equal, while a value of one (i.e., 100%) implies that the relaxation of the matching formulation is equal to the true objective of the optimal integral solution. The column "average" gives the average rgap, whereas column "maximal" shows the maximum relative closed gap for an instance of this sub test set.

For $n = 6$, there are few instances with a gap. However, for those instances for which there is a gap, it is clear that a sizable part of that gap is closed by the relaxation of the matching formulation. For larger values of $n$, many instances have a gap. We observe that a significant percentage of the gap is closed by the relaxation of the matching formulation. If $n$ is getting larger, however, both the value of the average gap closed as well as the value of the maximal gap closed decrease. We conclude that for small values of $n$, and thus for many realistic applications, the matching formulation provides a much better relaxation value than the traditional formulation.

### 5.3. A branch-and-price algorithm

Since the matching formulation can dominate the traditional formulation, a natural question is whether the stronger formulation also allows to solve the SRR problem faster than the traditional formulation. For this reason, we have implemented a branch-and-price algorithm (in the computational setup as described above) to compute optimal integral solutions of the matching formulation. That is, we use a branch-and-bound algorithm to solve the matching formulation, where each LP relaxation is solved using a column generation procedure.

*Implementation details.* In classical branch-and-bound algorithms, the most common way to implement the branching scheme is to select a variable $x_i$ whose value $x^\star_i$ in the current

**Table 1**
Comparison of the LP relaxation values of the traditional and matching formulation.

| | | All instances | | | | | Restricted to instances with $v^{LP}_{tra} < v^{IP}$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Average value | | | Solved | | | Average value | | | Solved | | Gap closed | |
| $n$ | $\rho$ | $v^{LP}_{tra}$ | $v^{LP}_{mat}$ | $v^{IP}$ | O | T | # | $v^{LP}_{tra}$ | $v^{LP}_{mat}$ | $v^{IP}$ | O | T | Average | Maximal |
| 6 | 0.5 | 2.227 | 2.297 | 2.380 | 50 | 0 | 14 | 2.452 | 2.702 | 3.000 | 14 | 0 | 43.45% | 100.00% |
| 6 | 0.6 | 3.802 | 3.865 | 3.920 | 50 | 0 | 12 | 3.924 | 4.188 | 4.417 | 12 | 0 | 52.08% | 100.00% |
| 6 | 0.7 | 5.430 | 5.510 | 5.540 | 50 | 0 | 9 | 5.611 | 6.056 | 6.222 | 9 | 0 | 66.67% | 100.00% |
| 6 | 0.8 | 7.620 | 7.635 | 7.660 | 50 | 0 | 4 | 7.250 | 7.438 | 7.750 | 4 | 0 | 37.50% | 100.00% |
| 6 | 0.9 | 10.003 | 10.040 | 10.060 | 50 | 0 | 6 | 10.361 | 10.667 | 10.833 | 6 | 0 | 66.67% | 100.00% |
| 12 | 0.5 | 0.080 | 0.080 | 0.080 | 50 | 0 | 0 | – | – | – | 0 | 0 | – | – |
| 12 | 0.6 | 2.018 | 2.213 | 3.480 | 50 | 0 | 49 | 2.019 | 2.217 | 3.510 | 49 | 0 | 15.29% | 100.00% |
| 12 | 0.7 | 8.022 | 8.342 | 9.500 | 50 | 0 | 50 | 8.022 | 8.342 | 9.500 | 50 | 0 | 22.27% | 70.77% |
| 12 | 0.8 | 17.184 | 17.474 | 18.340 | 50 | 0 | 50 | 17.184 | 17.474 | 18.340 | 50 | 0 | 29.89% | 100.00% |
| 12 | 0.9 | 31.459 | 31.654 | 31.840 | 50 | 0 | 31 | 31.096 | 31.410 | 31.710 | 31 | 0 | 56.87% | 100.00% |
| 18 | 0.5 | 0.000 | 0.000 | 0.000 | 50 | 0 | 0 | – | – | – | 0 | 0 | – | – |
| 18 | 0.6 | 0.060 | 0.060 | 0.060 | 50 | 0 | 0 | – | – | – | 0 | 0 | – | – |
| 18 | 0.7 | 2.045 | 2.292 | 5.600 | 50 | 0 | 50 | 2.045 | 2.292 | 5.600 | 50 | 0 | 6.68% | 15.66% |
| 18 | 0.8 | 19.831 | 20.330 | 23.900 | 50 | 0 | 50 | 19.831 | 20.330 | 23.900 | 50 | 0 | 12.37% | 28.19% |
| 18 | 0.9 | 52.700 | 53.066 | 54.500 | 50 | 0 | 49 | 52.673 | 53.047 | 54.510 | 49 | 0 | 21.55% | 100.00% |
| 24 | 0.5 | 0.000 | 0.000 | 0.000 | 50 | 0 | 0 | – | – | – | 0 | 0 | – | – |
| 24 | 0.6 | 0.000 | 0.000 | 0.000 | 50 | 0 | 0 | – | – | – | 0 | 0 | – | – |
| 24 | 0.7 | 0.200 | 0.200 | 4.340 | 0 | 50 | 49 | 0.163 | 0.163 | 4.408 | 0 | 49 | 0.00% | 0.00% |
| 24 | 0.8 | 12.352 | 12.893 | 24.180 | 0 | 50 | 50 | 12.352 | 12.893 | 24.180 | 0 | 50 | 4.57% | 7.91% |
| 24 | 0.9 | 69.327 | 69.922 | 74.860 | 29 | 21 | 50 | 69.327 | 69.922 | 74.860 | 29 | 21 | 10.69% | 21.68% |

LP solution is non-integral and to generate two subproblems by additionally enforcing either $x_i \leq \lfloor x^\star_i \rfloor$ or $x_i \geq \lceil x^\star \rceil$. In principle, this strategy is also feasible for the matching formulation, where the subproblems correspond to forbidding a schedule $M \in \mathfrak{M}$ for a round $r \in R$ or fixing the schedule in round $r$ to be $M$. This branching scheme, however, leads to a very unbalanced branch-and-bound tree as the former subproblem only rules out a very specific schedule, while the latter one fixes the matches of an entire round. Another difficulty of the classical scheme is that it might affect the structure of the pricing problem in the newly generated subproblems. Ideally, the pricing problem should not change such that the same algorithm can be used for adding new variables to the problem. We will address both issues next.

To obtain a more balanced branch-and-bound tree, we have implemented a custom branching rule following the Ryan–Foster branching scheme Ryan & Foster (1981): Our scheme selects a match $\{i, j\} \in \mathcal{M}$ in a round $r \in R$ and creates two children. In the left child, we forbid that $\{i, j\}$ is played in round $r$, and in the right child, we enforce that $\{i, j\}$ is played in round $r$. Note that for all matchings $M \in \mathfrak{M}$ this branching decision fixes all variables $y_{M,r}$ to zero if $\{i, j\} \in M$ for the left child, and $\{i, j\} \notin M$ for the right child.

Using this branching strategy, the structure of the pricing problem at each subproblem remains a matching problem. At the root node of the branch-and-bound tree, we need to solve a maximum weight perfect matching problem in a weighted version of $K_n$ as described above. At other nodes of the branch-and-bound tree, we have added branching decisions that enforce that two teams $i$ and $j$ either do meet or do not meet in a round $r \in R$. These decisions can easily be incorporated by deleting edges from $K_n$. When generating variables for round $r$, we remove edge $\{i, j\}$ from $K_n$ if $i$ and $j$ shall not meet in this round; if the match $\{i, j\}$ shall take place, then we remove all edges incident with $i$ and $j$ except for $\{i, j\}$. Consequently, our branching strategy allows to solve the LP relaxations of all subproblems in polynomial time.

Since our Python implementation of the traditional and matching formulation took too much time to be used in a branching scheme, we decided to implement our branch-and-price algorithm as a plug-in using the C-API of SCIP. The pricer plug-in is analogous, and maximal weight perfect matchings are now computed using the LEMON 1.3.1 graph library. To ensure that the branching decisions are taken into account, we also implemented a constraint that fixes $y_{M,r}$ to zero if the matching $M$ violates the branching decisions for round $r$, and added a plug-in that implements the branching decisions.

The branching rule sketched above admits some degrees of freedom in selecting the match $\{i, j\}$ and round $r$. In our implementation, we decided to mimic two well-known branching rules which are described in detail in the survey by Achterberg, Koch, & Martin (2005): most infeasible branching and strong branching on a selection of variables. Most infeasible branching branches on a binary variable with fractional value in an LP solution that is closest to 0.5, and strong branching branches on the variable that yields the largest dual bound improvement based on some metric. Since strong branching requires significant computational effort, it is common to make a limited branching candidate selection and apply strong branching on those.

The pseudo code for our branching rule is given in Algorithm 1. We start by computing the fractional match-in-round assignment values induced by the $y$-variables. Then, we make a selection of potentially good branching candidates $(m, r) \in \mathcal{M} \times R$, that is based on the $\text{score}_{m,r}$ metric shown in Line 8: this score prioritizes match-in-round assignments for which (i) the assignment value is close to 0.5, (ii) the cost coefficients are large, and (iii) the assignment values are relatively high. Using this score, we hope to resolve fractionality soon (by (i)). By (ii), we want to enforce a significant change of the objective value in the child that forbids match $m$, whereas the child enforcing that $m$ is played selects a match that is most likely played due to (iii). Experiments show that full strong branching leads to a smaller number of nodes to solve the problem, but this turns out to be very costly computationally. Therefore, we only apply strong branching for branch-and-bound tree nodes close to the root, and only evaluate a subset of branching candidates that have the highest $\text{score}_{m,r}$-metric. This is our candidate pre-selection. The higher the depth of the considered branch-and-bound tree node, the smaller the number of candidates considered. Of those branching candidates, we pick the candidate that maximizes $\text{score}^\star_{m,r}$ as defined in Line 18. The goal of this score is to choose the candidate where the objective values of the hypothetical children are different from the current node's objective. By considering the product of this difference, we prioritize if the objective of both hypothetical children have some difference with the current node objective. If the number of can-

**Algorithm 1:** Determining the branching candidate for an LP node.

**input** : An LP solution $y^\star_{m,r}$ in a branch-and-bound tree node atdepth $d$ with objective obj.
**output**: The branching decision(match $m \in \mathcal{M}$ in round $r \in R$),or detected integrality.

```
// fractional assignment of match m to round r
computeassign_{m,r} ← Σ_{M∈𝔐 : m∈M} y*_{M,r} for m ∈ M and r ∈ R;
if assign_{m,r} is 0 or 1for all m ∈ M and r ∈ R then
    return integral solution found;
// fractional part of assign_{m,r}
computefrac_{m,r} ← min{assign_{m,r}, 1.0 − assign_{m,r}}for m ∈ M and r ∈ R;
// score for every match-round pair
computescore_{m,r} ← frac_{m,r} · (1.0 + |c_{m,r}|) · (assign_{m,r})^2 for m ∈ M and r ∈ R;
// strong branching candidate selection
number_of_candidates ← max{1, ⌊0.1 · |M × R| · 0.65^d⌋};
if number_of_candidates > 1 then
    pick number_of_candidatescandidates (m, r) ∈ M × Rwith highest score_{m,r}as strong branching candidates;
    foreach strong branching candidate (m, r) do
        if score_{m,r} = 0.0 then
            // then assign_{m,r} is 0 or 1, skip this candidate
            continue (i.e., skip this candidate);
        apply strong branching on (m, r),with objectivesobj_forbid, obj_enforcein the two children;
        compute score*_{m,r} ←
        (obj_forbid − obj + 1.0) · (obj_enforce − obj + 1.0);
    return branch on strong branching candidate (m, r)with maximal~score*_{m,r};
else
    // do not apply strong branching deep in the branch-and-bound tree
    return branch on (m, r) ∈ M × Rwith maximal score_{m,r};
```

didates is only one, no strong branching is applied and that candidate match-in-round assignment is chosen for branching.

All experiments have been run on a Linux cluster with Intel Xeon E5 3.5 gigahertz quad core processors and 32 gigabyte memory. The code was executed using a single thread and the time limit for all computations was 2 hours per instance.

*Numerical results*. Table 2 summarizes our results for our instances for $n \in \{6, 12, 18\}$. We distinguish the instances by their parameters $n$ and $\rho$, and we report on the number of instances that could be solved (resp. could not be solved) within the time limit in column "O" (resp. "T"). Moreover, we report on the minimum, mean, and maximum running time per parameterization. The mean of all running times $t_i$ is reported in shifted geometric mean $\prod_{i=1}^{50}(t_i + s)^{\frac{1}{50}} - s$ using a shift of 10 seconds to reduce the impact of instances with very small running times.

We observe that instances with 6 and 12 teams can be solved very efficiently within fractions of seconds in the former and within seconds in the latter case. Instances with 18 teams are more challenging, in particular, if the ratio $\rho \in \{0.7, 0.8\}$. In this case, only 3 and 22 instances could be solved, respectively, but note that not all instances are equally difficult. For instance, for $n = 18$ and $\rho = 0.8$, there exists an instance that can be solved within roughly five minutes, whereas the mean running time is more than an hour. To fully benefit from the strong LP relaxation of the matching formulation, it might be the case that additional algorithmic enhancements can further improve the performance of the branch-and-price algorithm.

**Table 2**
Computational results for the branch-and-price algorithm for Model (M).

| | | | Solved | | Solving time (seconds) | | |
|---|---|---|---|---|---|---|---|
| $n$ | $\rho$ | # | O | T | min | mean | max |
| 6 | 0.5 | 50 | 50 | 0 | 0.00 | 0.00 | 0.01 |
| 6 | 0.6 | 50 | 50 | 0 | 0.00 | 0.00 | 0.01 |
| 6 | 0.7 | 50 | 50 | 0 | 0.00 | 0.00 | 0.01 |
| 6 | 0.8 | 50 | 50 | 0 | 0.00 | 0.00 | 0.01 |
| 6 | 0.9 | 50 | 50 | 0 | 0.00 | 0.00 | 0.01 |
| 12 | 0.5 | 50 | 50 | 0 | 1.25 | 2.38 | 5.73 |
| 12 | 0.6 | 50 | 50 | 0 | 0.12 | 4.09 | 8.28 |
| 12 | 0.7 | 50 | 50 | 0 | 0.21 | 3.33 | 7.38 |
| 12 | 0.8 | 50 | 50 | 0 | 0.14 | 2.05 | 7.63 |
| 12 | 0.9 | 50 | 50 | 0 | 0.11 | 0.54 | 2.21 |
| 18 | 0.5 | 50 | 50 | 0 | 54.61 | 106.09 | 210.77 |
| 18 | 0.6 | 50 | 48 | 2 | 103.41 | 866.21 | 7200.00 |
| 18 | 0.7 | 50 | 3 | 47 | 930.53 | 6854.47 | 7200.09 |
| 18 | 0.8 | 50 | 22 | 28 | 312.65 | 4084.21 | 7200.06 |
| 18 | 0.9 | 50 | 50 | 0 | 6.74 | 332.98 | 3990.61 |

## 6. Conclusion

The use of integer programming for finding schedules of round robin tournaments is widespread. We have compared three formulations for this problem, one of which (the matching formulation) is stronger than the other formulations. We have proposed a class of valid inequalities for the matching formulation, which may be of use when developing cutting-plane based techniques for this problem. By randomly generating instances, we studied the strength of the formulations, and we implemented a branch-and-price algorithm based on the matching formulation to see its efficiency. Although this algorithm is able to solve small-scale instances rather efficiently, solving large instances of the SRR efficiently remains a challenge.

It is a relatively straightforward exercise to generalize the formulations in this paper to $k$-Round Robin tournaments for $k \geq 2$. In such a setting, additional properties may become relevant; we mention home-away assignments and whether or not a tournament should be split into $k$ parts, where each pair of teams meets once in each part. It is a fact that most of the theoretical results derived in the previous section continue to hold for these situations, see van Doornmalen et al. (2022) for more details.

Possible directions of future research are to further strengthen our integer programming formulations/techniques. On the one hand, once can investigate additional cutting planes to strengthen both the traditional and matching formulation. For the matching formulation, cutting planes will in particular affect the pricing problem and thus might change its structure. Thus, the trade-off between the strength of cutting planes and the difficulty of solving the pricing problem that needs to be investigated. On the other hand, one can enhance our branch-and-price algorithm in several directions, e.g., the development of more sophisticated branching rules or heuristics for producing good schedules.

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