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Theory and Methodology

Geometric three-dimensional assignment problems ★

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Abstract

We investigate two geometric special cases of the three-dimensional assignment problem: Given are three sets B, R and G (blue, red and green) each containing n grid points in the Euclidean plane. We want to find a partition of $B \cup R \cup G$ into n three-colored triangles such that (a) the total circumference of all triangles or (b) the total area of all triangles becomes minimum. Both versions of the problem are proved to be NP-hard.

Keywords: Three-dimensional assignment problems; Computational complexity

1. Introduction

The problem and some of its history

The general three-dimensional assignment problem (3AP) is one of the most famous 'basic' problems in combinatorial optimization and has been actively investigated in the literature. It is defined as follows: There are given three sets B, R and G (a blue, a red and a green set) each with n elements, and a cost function $c: B \times R \times G \to \mathbb{R}^+$. The goal is to partition $B \cup R \cup G$ into n three-colored triples $t_i = (b_{j(i)}, r_{k(i)}, g_{l(i)})$ such that $\sum_{i=1}^{n} c(t_i)$ becomes minimum.

The problem is well known to be NP-hard, see Karp [13]. Frieze [9] gave a bilinear programming formulation. Branch and bound methods are due to Vlach [20] and Pierskalla [15]. Hansen and Kaufman [12] designed a primal-dual implicit enumeration method based on a graph theoretic approach. Fröhlich [10] and Burkard and Fröhlich [4] improved the bounding technique by using subgradient optimization techniques. Recently, Balas and Saltzman [1,2] investigated the facial structure of 3AP.

Special cases

Since the general 3AP is NP-hard (and even hard to approximate), the computational complexity of special cases is of interest.

Bein et al. [3] have shown that d-dimensional assignment problems can be solved in polynomial time, if the corresponding cost-array possesses the Monge property; this is essentially the only known special case that is solvable in polynomial time. On the other (NP-hard) hand, Crama and Spieksma [7] considered 3AP's where the cost function fulfills some special triangle inequality. Although this triangle inequality makes the problem easy to approximate ([7] gives an approximation algorithm with worst-case ratio 4/3), it does not remove the NP-hardness from the problem. Finally, Burkard, Rudolf and Woeginger [5] investi-

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gated the special case of decomposable cost functions and derived another NP-completeness result.

Our results

In this note, we prove NP-hardness for two geometric special cases of the 3AP. In our cases, the elements of B, R and G are grid points (points with integer coordinates) in the Euclidean plane, and the cost function c assigns to a triple of points either (a) the circumference or (b) the area of the corresponding triangle (clearly, the circumference cost function fulfills the triangle inequality conditions of [7]). These two versions of the problem will be abbreviated by C3AP and A3AP, respectively.

Our results are not surprising (as the 3AP is a *very hard* problem), but they are tedious to derive. We mainly fight against irrational coordinates, irrational distances and similar things. For example, we just claim NP-hardness (instead of NP-completeness) since C3AP is not known to be in NP.

The NP-hardness result we derive here for C3AP contrasts with the polynomial solvability of this problem when restricted to a line (notice that A3AP becomes trivial on a line). Indeed, when all points lie on a line, a result in Queyranne et al. [17] implies that a simple "from left to right" algorithm solves C3AP.

Motivation

Consider the following situation arising in the assembly of printed circuit boards. A robot-arm inserts, on given locations on a printed circuit board, electronic components. The robot arm described in e.g. [6,19] is equipped with three heads, which implies that at most three locations can be visited in one so-called placement round. Moreover, since each head has some specific equipment to handle the components, the set of locations is partitioned into three mutually disjunct subsets, one for each head. Let us refer to the set of locations visited in a placement round as a triple. In order to minimize the distance travelled by the arm (thus maximizing the throughput rate), it is important to construct triples whose locations are, in some sense, close to each other. It is not difficult to see that, when one chooses the circumference or the area of a triple as criterion, this problem is equivalent to C3AP or A3AP respectively. For a more elaborate description of this application, see [6,19].

Another application arises in the field of multitarget tracking and can be described as follows. N objects move along straight lines in the plane. At K time-instants a scan is made and the position of each object is observed and recorded. From such a scan it is not possible to deduce which object generates which observation. Also, an error may be associated to each observation. The problem is now to identify N K-tuples of observations, called tracks, such that each track contains one observation from each scan, while minimizing a least squares criterion (see Poore and Rijavec [16] for a more extensive description). This problem is called the data-association problem and, if $K = 3$, the decision version of this problem reduces to A3AP.

Organization of the paper

Section 2 contains some preliminary definitions, propositions and results. In Section 3, we give the NP-hardness proof for the circumference problem C3AP, and in Section 4 we derive the corresponding result for A3AP.

2. Preliminaries

In our first NP-hardness proof, we will make use of so-called *rectilinear planar layouts* of planar graphs. A *rectilinear planar layout* of a planar graph $G = (V, E)$ maps the vertices in V to horizontal line segments and the edges of G to vertical line segments, with all end points of segments at positive integer coordinates. Two horizontal segments are connected by a vertical segment if and only if the corresponding vertices are adjacent in the graph. The following proposition is due to Rosenstiehl and Tarjan [18].

Proposition 2.1. *Given a planar graph $G = (V, E)$ with n vertices, then a rectilinear planar layout of G can be computed in $O(n)$ time. Moreover, the height and the width of the layout are both $O(n)$. Without loss of generality we may assume that all horizontal segments are at different (integer) heights and that all vertical segments have pairwise distinct x-coordinates.*

The following problem is a very special version of the 3-dimensional matching problem (see Garey and Johnson [11]). Both of our reductions will be done to this problem.

Planar 3-dimensional matching (P3DM)

Input. Three pairwise disjoint sets X, Y and Z with $|X| = |Y| = |Z| = q$ and a set $T \subseteq X \times Y \times Z$ such that

(i) every element of $X \cup Y \cup Z$ occurs in at most three triples from T, and such that

(ii) the induced graph G is planar. (This induced graph G is defined as follows: It contains a vertex for every element of $X \cup Y \cup Z$ and for every triple in T. There is an edge connecting a triple to an element if and only if the element is a member of the triple. Clearly, G is bipartite with vertex bipartition $X \cup Y \cup Z$ and T).

Question. Does there exist a subset T' of q triples in T such that each element of $X \cup Y \cup Z$ is contained in precisely one triple from T' ?

Proposition 2.2 (Dyer and Frieze [8]). *The problem* P3DM *is NP-complete.*

3. Minimizing the circumference

In this section, we will prove that C3AP is NP-hard. Since distances computed via the Euclidean metric may be irrational, it is not known how to evaluate the cost of some feasible solution to an instance of problem C3AP in polynomial time. Thus, C3AP is not known to be in NP.

The main idea of the proof given below can also be found in Pferschy et al. [14], where it is shown that the problem clustering into triangles (given $3m$ points in the plane, find m triangles such that the total circumference is minimal) is NP-hard. Here, due to the presence of colors, we have to be more careful in our arguments, and use some additional gadgets.

Theorem 3.1. *For three sets B, R, G (with $|B| = |R| = |G| = n$) of points in the Euclidean plane, it is NP-hard to decide whether there exists a partition into three-colored triangles with total circumference at most $12n$.*

Proof. Let X, Y, Z and $T \subseteq X \times Y \times Z$ (with $|X| = |Y| = |Z| = q$ and $|T| = t$) constitute an instance of P3DM. We will construct a point set $P = B \cup R \cup G$, with $|B| = |R| = |G| = n$ (where the value of n will be determined later) that allow a three-colored triangle partition with total circumference at most $12n$ iff the P3DM-instance has answer YES.

In the proof we will make abundant use of a right-angled triangle with side lengths 3, 4 and 5; it will allow us to keep all points in P at integer coordinates. Let us denote this triangle by Δ. Consider the planar graph G induced by the instance of P3DM. We compute a rectilinear planar layout of G (see Section 2) and multiply all coordinates by 1000.

Now, for every element of $X \cup Y \cup Z$, P will contain one so-called *element point*. And for every triple in T, P contains three points forming a copy of Δ. We will refer to a Δ which corresponds to a triple in T as a *triple triangle*. The element points and the triple triangles are placed somewhere at the corresponding line segments in the planar rectilinear layout (at integer coordinates, of course). To further construct the instance of C3AP, we are going to build paths, consisting of copies of Δ, which connects an element point to some point from its corresponding triple triangle. Such a path will roughly follow the line segments which connect the corresponding two points in the rectilinear planar layout. To demonstrate how this construction works, we are going to show three things:

(i) how (at most) three paths leave an element point,

(ii) how paths "come together" at a triple triangle, and

(iii) what building blocks are used to construct a path.

Of course, we also have to specify the color of each point we use in the construction. This will be done informally by using the symbol R, B or G for a red, a blue or a green point respectively. Obviously, the color of an element point is red, blue or green if this point corresponds to an element from X, Y or Z respectively.

Ad (i) Consider an element point corresponding to $x_i \in X$, $1 \leqslant i \leqslant q$. This point is connected to (at most) three paths, each leading to a different triple triangle. How these paths start in an element point is depicted in Fig. 1.

Notice that no feasible triangle with circumference < 12 exists (although an "unusual" triangle occurs, namely (R_4, B_6, G_3); later we will argue that this triangle cannot occur in a solution with value $12n$). Also, notice that the colors green and blue are forced in the sense that it is not possible to interchange for instance B_4 and G_4 without producing feasible triangles with circumference < 12.

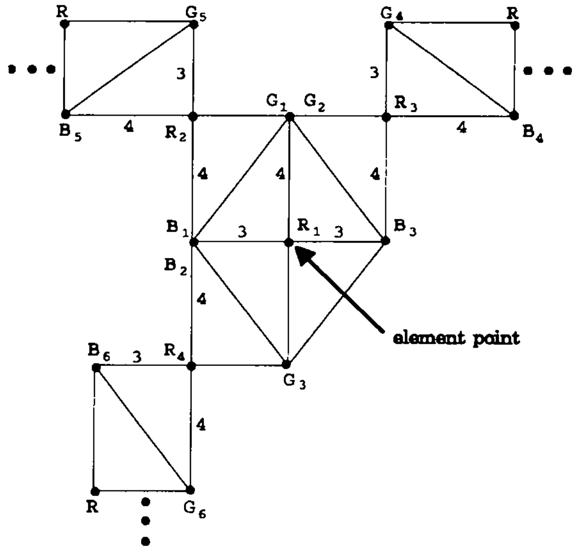

Fig. 1.

Ad (ii) Consider a triple triangle with points (t_1, t_2, t_3). The three paths which come together in this triple triangle origin from three mutually different colored element points. Now, each t_i, $1 \leqslant i \leqslant 3$, will serve as "end point" (or target point) of one path. The color of t_i, $1 \leqslant i \leqslant 3$, will be identical to the color of the element point where the path, ending at t_i, started. See Fig. 2, where we assumed, without loss of generality, that t_1, t_2 and t_3 are colored R, B and G respectively.

Notice that no feasible triangle with circumference < 12 exists, and that the coloring of the points is forced, given the colors of t_1, t_2 and t_3.

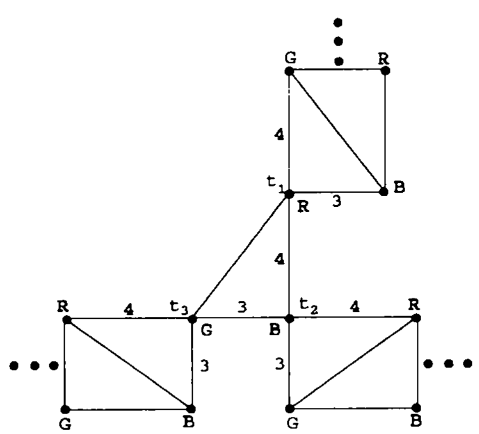

Fig. 2.

Ad (iii) To construct a path connecting an element point and a point from a triple triangle, we need to be able to travel horizontally, to travel vertically and to change direction from travelling horizontally to vertically or vice versa. In Fig. 3, we depict ways how this can be achieved. (We will assume that we are dealing with a "red" path, that is a path whose element point is red).

Using the building blocks so far, we can come close to connecting an element point to a point from a triple triangle, but we have to argue that we can make a precise connection. Now suppose that we are dealing with a red path, and let us denote the point from the triple triangle to which the path has to be connected as the target point. It is not difficult to verify that the building blocks from Fig. 3 allow one to arrive at an integer point with a deviation from the target point bounded by (4,4). (This value is not tight, but sufficient for our purpose).

Now, we will introduce a gadget which can shift a horizontal segment of the path one unit upwards (or downwards) without changing the x-coordinate of the current end point of the horizontal path. Moreover, we can apply this gadget, mutatis mutandis, to a vertical segment, that is we can change the x-coordinate of the current end point of the path with one unit without changing the y-coordinate. See Fig. 4.

Notice that no triangle with circumference < 12 exists; however, in Fig. 4 "unusual" triangles with circumference 12 do occur. Later we will argue that these triangles cannot be present in a solution of C3AP with value $12n$.

Of course, by using this gadget at most 4 times in the horizontal direction and at most 4 times in the vertical direction, we can precisely connect the path to the target point. The multiplication by 1000 ensures that there is enough room available to accommodate this gadget.

Finally, we have to deal with the following difficulty (again assume we are dealing with a red path). We have shown how to connect an element point and a point from a triple triangle, but it is conceivable that the colors blue and green in this connection do not match. Since the colorings leaving an element point and a triple triangle point are forced (see Figs. 1 and 2), we need another gadget to be able to "interchange" the colors blue and green in a red path. See Fig. 5.

Notice that this gadget indeed induces a switch of

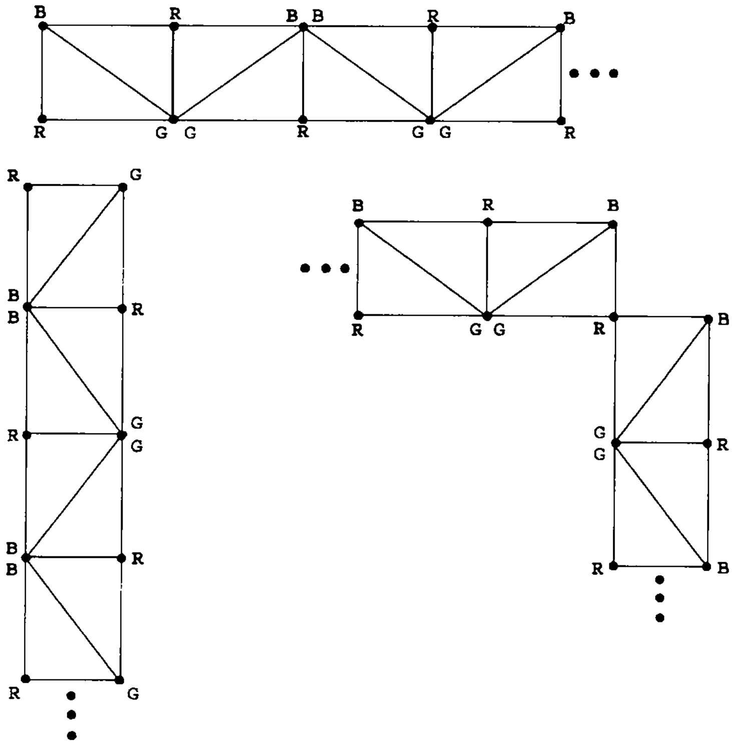

Fig. 3.

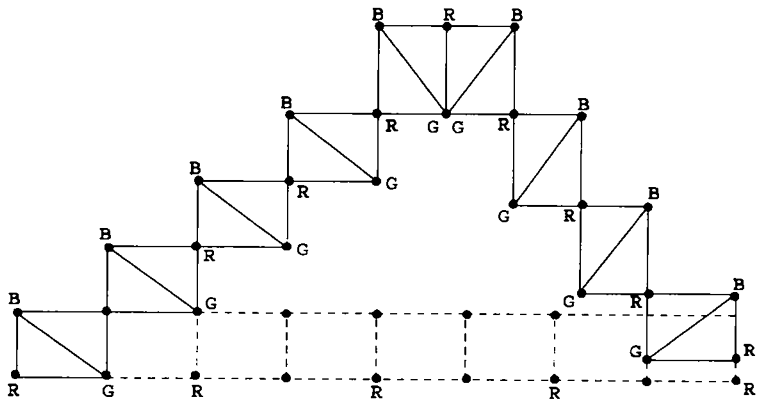

Fig. 4.

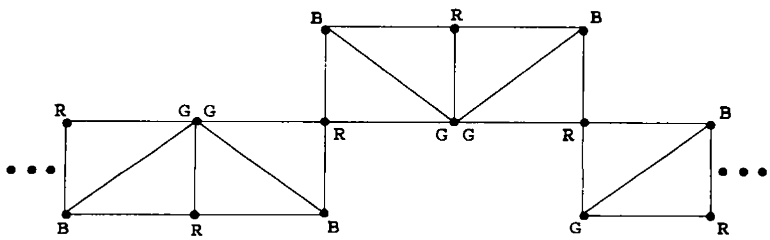

Fig. 5.

the colors blue and green. Also, notice that three "unusual" triangles with circumference 12 occur.

By using the constructions described, it is possible to produce an instance of C3AP. Now, we must argue that a solution of value $12n$ corresponds to a feasible solution of P3DM and vice versa.

(If) Suppose a clustering into triangles with mutually different colored points with value $12n$ exists. Since there is no feasible triangle in the construction with value < 12, it follows that each triangle has circumference 12. Consider element point R_1 in Fig. 1. It must be in a triangle with circumference 12. There are four distinct possibilities: (R_1, B_1, G_1), (R_1, B_3, G_1), (R_1, B_3, G_3) and (R_1, B_1, G_3). However, (R_1, B_3, G_3) can be excluded, since in that case one can verify by inspection that no feasible triangles with circumference 12 exist to accommodate the points G_1, G_2, B_1 and B_2.

Case 1: suppose (R_1, B_3, G_1) is in the solution. Consider G_2. There are two red points with which it can be in a triangle of circumference 12, namely R_2 and R_3. However, R_3 is impossible since there is no blue point giving a feasible triangle with value 12. (B_3 is already matched in this case.) Thus we get the triangles (R_2, B_1, G_2) and (R_3, B_4, G_4) and then it is not hard to argue that the only way to find a triangle for B_2 is (R_4, B_2, G_3) which must be in the solution then. Informally, we will refer to this case as the case where the element point belongs to the North-East path.

Case 2: suppose (R_1, B_1, G_1) is in the solution. How to match B_3? The only possibility is (R_3, B_3, G_2) and similarly for G_3 we find (R_4, B_2, G_3). Thus (R_2, B_5, G_5) also must be in the solution. Here, the element point belongs to the North-West path.

Case 3: suppose (R_1, B_1, G_3) is in the solution. Arguments similar to those used in the previous cases yield that (R_2, B_2, G_1), (R_3, B_3, G_2) and (R_4, B_6, G_6) must be in the solution. The element point belongs to the South-West path.

Consider a path. There are precisely two distinct possibilities of finding feasible triangles with circumference 12 in a path, see Fig. 6.

This also holds for the gadgets introduced as is easily verified by the reader. Which of the two possibilities is the case depends solely on the beginning of the path, which is completely described by the cases 1, 2 and 3. So, for each path there are two possibilities: either the element point belongs to the path (in which

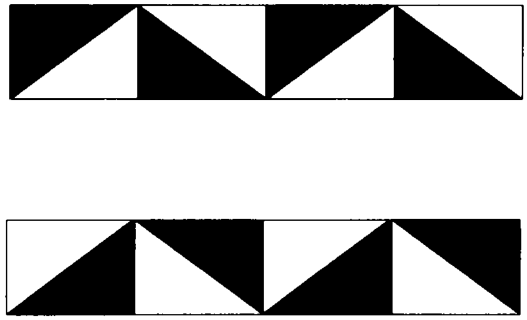

Fig. 6.

case we will argue that the target point to which the path is connected does not belong to the path), or it does not belong to the path (in which case the target point does).

The number of triangles in a path is even. If the element point belongs to the path, triangles $1, 3, 5, \ldots, 2\alpha - 1$ are in the path, and hence the target point from the triple triangle is not. How to find a feasible triangle with circumference 12 for this point? The only candidate is the triple triangle itself. But then it follows that the element points corresponding to the other two paths arriving at this triple triangle, also belong to that path.

Thus, in a solution with value $12n$, each element point is assigned to precisely one triple triangle, and if a triple triangle has an element point assigned to it, the other two element points (which all have different colors) must also be assigned to it. This gives a solution to P3DM.

(Only if) Trivial.

Since our construction obviously can be performed in polynomial time, this argument completes the proof of the theorem. □

Theorem 3.2. *Problem* C3AP *is NP-hard.*

Remark. As mentioned in the introduction, C3AP is a special case of a problem treated in [7]. Therefore, the approximation algorithms proposed therein with the corresponding worst-case ratio's, remain valid for C3AP. Thus, by applying these algorithms, one can find in polynomial time, solutions to C3AP whose cost is bounded by 4/3 times the cost of an optimal solution. In fact, since the example for which this ratio is achieved is not Euclidean [7], it is conceivable that a better bound is possible.

4. Minimizing the area

In this section, we deal with the computational complexity of problem A3AP. We will see that this problem is much easier to handle than problem C3AP. First of all, the decision version of A3AP lies in NP: Just guess any partition into three-colored triangles, compute the corresponding total area and compare it to the given bound. The crucial fact is that the area of a triangle with grid points as vertices is a *rational* number (either it is an integer or an odd integer divided by two) and easy to compute. Hence, the computation and summing of all areas can be performed within time that is polynomial in the decimal representation of the coordinates. This shows that A3AP belongs to NP. Secondly, the NP-hardness proof is conceptually easier to derive.

Theorem 4.1. *For three sets B, R, G ($|B| = |R| = |G|$) of points in the Euclidean plane, it is NP-complete to decide whether there exists a partition into three-colored triangles with total area zero.*

Proof. We show the NP-completeness by reducing problem P3DM to it (cf. Section 2). Let X, Y, Z and $T \subseteq X \times Y \times Z$ (with $|X| = |Y| = |Z| = q$ and $|T| = t$) constitute an instance of P3DM. We will construct three point sets B, R and G that allow a three-colored triangle partition with total area zero iff the P3DM-instance has answer YES.

All grid points introduced in our construction will have x-coordinates between 1 and $3q$ (to every element of $X \cup Y \cup Z$ we assign a unique x-coordinate in this range). There will be $3t$ so-called *triple points* with positive y-coordinates between 1 and $14t^2$. Moreover, there will be up to $12q$ so-called *element points* with negative y-coordinates between -1 and $-(3t+12q)^2$.

Since a triangle with area zero is a line segment, our main problem consists in avoiding 'unallowed' collinearities. We will ensure that the only collinearities in our point sets will be vertical or horizontal. Our construction is done in two phases.

In the first phase, we introduce the $3t$ triple points as follows. We make t steps, each step corresponding to a triple, and we introduce 3 points per triple, one point per color. Suppose, we already treated $(i-1)$ triples and now have to deal with the ith triple, say triple (x_i, y_i, z_i). We will introduce three new points, one for every x-coordinate assigned to x_i, y_i and z_i, respectively. All three points will have the same y-coordinate. Moreover, the new points will not form a triangle of area zero with any two of the $3i-3$ old points introduced till now. This can be reached as follows: The $3i-3$ old points determine at most $(3t-3)(3t-4)/2$ lines, and these lines intersect the three vertical lines corresponding to x_i, y_i and z_i in at most $3(3t-3)(3t-4)/2 < 14t^2 - 1$ points. Hence, there exists at least one integer y^* in the range $[1, 14t^2]$ that does not collide with any of these lines. We put the three new points at this y-coordinate y^*. The point corresponding to x_i is colored blue, the point corresponding to y_i is colored red, and the point corresponding to z_i is colored green. Then we move on to the next step.

In the second phase, we introduce the $\leqslant 12q$ element points. We go through $3q$ steps, every step corresponding to some element in $X \cup Y \cup Z$. In every step, we introduce up to four new points colored by the two complementary colors: Every element occurs in two or three triples of T. In case it occurs in two triples, we introduce two new points and in case it occurs in three triples, then we introduce four new points. There are two colors not corresponding to the treated element. Half of the new points are colored by one of these colors, half are colored by the other one (e.g. if the element is in X and occurs in three triples, then we introduce two blue and two green points for it). The new points will get the x-coordinate that is assigned to the corresponding element. To avoid collinearities to the outside world, we must take care that the new points do not form triangles of area zero with old points. However, in the jth step, there are $3t$ triangle points from the first phase and at most $4(j-1) < 12q-4$ element points. If we construct all lines determined by these points and intersect them with the vertical line under consideration, then there remain at least four integer y-coordinates in the range $[-(3t+12q)^2, -1]$ that are not covered by these lines. We put the four (two) new points at four (two) of these coordinates and assign appropriate colors to them. This completes the construction of the A3AP-instance.

It remains to show that A3AP has a solution with total area zero iff P3DM possesses a solution.

(If) In case P3DM has a solution, say the triples $t_{i_1}, \ldots, t_{i_q}$, we take the triple points introduced in steps

$i_1, \ldots, i_q$ during the first phase. These points form q horizontal line segments, i.e. triangles of area zero. This leaves on every vertical line one or two uncovered triple points in the corresponding color and two or four uncovered element points in the complementary colors. We group these points in the obvious way to produce a set of vertical triangles of area zero that cover the remaining points.

(Only if) In case A3AP has a solution with total area zero, consider the way the element points are assigned to the triangles (i.e. segments): The only possible partners are the points on the same vertical line (as by our construction, no other collinearities are possible). Hence, they will belong to vertical triangles, and on each vertical line, there remains a single uncovered triple-point. Within the set of these uncovered triple points, the only remaining collinearities are horizontal; each such collinearity corresponds to some triple in T. Now since all points must belong to some segment, we have detected a solution to P3DM.

Since our construction obviously can be performed in polynomial time, this argument completes the proof of the theorem. □

Theorem 4.2. *Problem* A3AP *is NP-complete.*

Remark. A similar construction as in the proof of Theorem 4.1 can be used to show that finding a set of covering segments with minimum total length is NP-hard (i.e. we do not only require the total area to be zero, but also that the total circumference of the triangles of area zero becomes minimum).

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