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Approximation algorithms for multi-index transportation problems with decomposable costs

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Abstract

The axial multi-index transportation problem is defined as follows. Given are k sets A_r, each set having n_r elements, $r = 1,\ldots,k$. The cartesian product of the sets A_r is denoted by A. To each element $a \in A$ a certain cost, $c_a \in \mathbb{R}$, is associated. Further, a nonnegative demand e_{ri} is associated to each set $A_{ri} = \{a \in A : a(r) = i\}$. The problem is to find nonnegative real numbers x_a such that each demand is satisfied (that is $\sum_{a \in A_{ri}} x_a = e_{ri}$ for $r = 1,\ldots,k,\ i = 1,\ldots,n_r$) and such that total cost (that is $\sum_{a\in A} c_a \cdot x_a$) is minimized.

In this paper we deal with a special case of this problem where the costs c_a are *decomposable*, that is, given a real-valued function f and a distance $d_{ij}^{A_r \times A_s}$ between element i of A_r and element j of A_s, we assume that $c_a = f(d_{a(1),a(2)}^{A_1 \times A_2},\ldots,d_{a(k-1),a(k)}^{A_{k-1}\times A_k})$ for all $a \in A$. We present two algorithms for this problem, and we analyze their worst-case behavior without requiring explicit knowledge of the cost-function f. Next, we use these results to derive explicit bounds in the case where f is the diameter cost-function (that is $c_a = \max_{r,s} d_{a(r),a(s)}^{A_r \times A_s}$), and in the case where f is the Hamiltonian path cost-function (that is $c_a = \min\{\sum_{i=1}^{k-1} d_{a(\sigma(i)),a(\sigma(i+1))}^{A_{\sigma(i)}\times A_{\sigma(i+1)}} : \sigma$ is a cyclic permutation of $\{1,\ldots,k\}$).

1. Introduction

The *axial multi-index transportation problem* can be formulated as follows. Given are k sets A_r with $A_r \doteq \{1,\ldots,n_r\}$, for $r \in K$ with $K \doteq \{1,\ldots,k\}$. The cartesian product of these sets A_r, $1 \leqslant r \leqslant k$, is denoted as A, that is $A \doteq \{a : a \in A_1 \times A_2 \times \cdots \times A_k\}$. We will refer to an element $a \in A$ as *cluster* a, with $a(r)$ denoting the

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r-th entry of vector a. For each cluster $a \in A$ a certain *cost* $c_a \in \mathbb{R}$ is specified. Further, for $r \in K$, $i \in A_r$, we define a *section* $A_{ri} \doteq \{a \in A : a(r) = i\}$. To each section A_{ri}, a nonnegative *demand* $e_{ri} \in \mathbb{R}$ is associated. The problem is now to find nonnegative real numbers x_a, $a \in A$, such that the sum of those numbers x_a for which $a(r) = i$, is equal to e_{ri} for each $r \in K$, $i \in A_r$, and such that total cost, summed over all clusters, is minimized. Mathematically, the problem can be described as follows:

$$(kTP) \qquad \text{minimize} \quad \sum_{a \in A} c_a x_a$$

$$\text{such that} \quad \sum_{a \in A_{ri}} x_a = e_{ri} \quad \text{for } r \in K;\ i \in A_r,$$

$$x_a \geqslant 0 \quad \text{for } a \in A.$$

We assume that $\sum_{i \in A_r} e_{ri} = \sum_{j \in A_s} e_{sj}$ for all $r, s \in K$. It is not difficult to show that this assumption is a necessary and sufficient condition for the existence of a feasible solution to (kTP). Notice that for $k = 2$ the familiar (2-index) transportation problem arises.

Axial k-index transportation problems have not been widely studied for $k \geqslant 3$. An early reference to the axial 3-index transportation problem $(3TP)$ is [14]. Other early references to $(3TP)$ are [8, 3, 4]. More recently, in [13], the special case of (kTP) is investigated, where all elements of the sets A_r, $r \in K$, lie on a single line, and the cost of each cluster a depends on the distances between elements of that cluster.

A problem closely related to (kTP) is the axial k-index *assignment* problem. This problem arises when the sets A_r have equal size, all demands e_{ri} are equal to 1, and x_a is restricted to be either 0 or 1 for all $a \in A$ (see [12]). Multi-index assignment problems occur in various real-world situations (see [1] for a recent overview). For instance, the scheduling of classes, teachers and rooms (see [6]) as well as the manufacturing of printed circuit boards (see [5]), may give rise to (specially structured) instances of the 3-index assignment problem. Another interesting application of a problem related to the k-index assignment problem can be found in computational molecular biology (see [7, 11]).

This paper deals with the special case of (kTP) where the costs c_a, $a \in A$, are not arbitrarily given numbers, but are in some sense decomposable. More precisely, we will assume that there exist for each pair of sets A_r, A_s, $r \neq s$, nonnegative numbers $d_{ij}^{A_r \times A_s}$, representing a *distance* between each element i of A_r and each element j of A_s, which determine the cost of a cluster $a \in A$. Formally, the costs c_a, $a \in A$, are defined to be *decomposable* if there exists a function $f : \mathbb{R}^{\binom{k}{2}} \to \mathbb{R}$ (called the cost function) such that

$$c_a = f\left(d_{a(r),a(s)}^{A_r \times A_s}\right), \quad r, s \in K, \ r \neq s, \qquad \text{for all } a \in A.$$

We will further assume that the distance (or length) function d is *symmetric*, that is,

$$d_{ij}^{A_r \times A_s} = d_{ji}^{A_s \times A_r} \quad \text{for all } r,s \in K,\ i \in A_r,\ j \in A_s.$$

The motivation for investigating decomposable costs is that often, in practical applications, some structure in the cost-coefficients c_a can be found. A potential way to capture this structure is by using the concept of decomposable costs. Indeed, in a number of applications these decomposable costs arise naturally (see [5–7]. In [5] it is shown that for $k \geqslant 3$, and for some simple, decomposable cost functions the k-index *assignment* problem is $\mathcal{NP}$-hard.

Bandelt et al. [2] present heuristics for the multi-index *assignment* problem with decomposable costs, along with worst-case analyses for different specifications of the cost-function (see also [7]). For instance, in the case where the cost of a cluster is equal to the sum of all distances in the cluster, Bandelt et al. [2] propose an algorithm which is guaranteed to find a solution with a cost bounded by twice the cost of an optimal solution for arbitrary $k \geqslant 2$. These worst-case analyses depend on the assumption that d satisfies the triangle inequality (see Section 4).

The purpose of this paper is to present a general, unifying framework for such worst-case analyses. The new contributions are (i) an extension to multi-index transportation problems; (ii) an elucidation of the role of the triangle inequality in deriving such results; and (iii) the treatment of new cost functions, the diameter and shortest Hamiltonian path. In the next section, the heuristics are introduced, and in Section 3 a general worst-case analysis is presented. Finally, in Section 4, two specific cost functions are investigated.

2. Single-Hub and Multiple-Hub Heuristics

First, consider the following two-step heuristic defined for a fixed index h. The first step amounts to solving $k-1$ ordinary (that is, 2-index) transportation problems with respect to A_h and A_r for all $r \in K$, $r \neq h$. In the second step, a solution to the k-index transportation problem is constructed based on the solutions found in the first step. We will refer to the heuristic as the *Single-Hub* heuristic (cf. [2]). A formal description is as follows, where we denote an optimal solution to an (ordinary) 2-index transportation problem defined by demands e_{ri} and e_{sj} and distance function $d^{A_r \times A_s}$, by $y^{A_r \times A_s}$ for some $r,s \in K$ with $r \neq s$. The solution to the k-index transportation problem is given by $\{z_a^h\}$.

Step 1 Fix h, $1 \leqslant h \leqslant k$. For all $r \in K \setminus \{h\}$, compute $y^{A_r \times A_h}$. Set $\tilde{y}^{A_r \times A_h} = y^{A_r \times A_h}$ for $r \in K \setminus \{h\}$.
Step 2 For $j := 1$ to n_h do
begin

$q := 0$
$a(h) := j$

$$a(i) := 1 \text{ for all } i \in K \setminus \{h\}$$

inner loop:

while $q < e_{h,a(h)}$ do
begin
  let ℓ be such that $\tilde{y}^{A_\ell \times A_h}_{a(\ell),a(h)} = \min_{r \neq h} \tilde{y}^{A_r \times A_h}_{a(r),a(h)}$
  $z^h_a := \tilde{y}^{A_\ell \times A_h}_{a(\ell),a(h)}$
  $\tilde{y}^{A_r \times A_h}_{a(r),a(h)} := \tilde{y}^{A_r \times A_h}_{a(r),a(h)} - z^h_a$ for all $r \in K \setminus \{h\}$
  $a(\ell) := a(\ell) + 1$
  $q := q + z^h_a$
end;

end.

We now illustrate the algorithm on an instance of $(3TP)$ as presented in the following example. The feasibility of the algorithm is given later in Theorem 2.1.

Example 2.1. Let $k = 3$, $n_1 = n_2 = 3$, $n_3 = 4$,

$$d^{A_1 \times A_2} = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 5 & 6 \\ 8 & 4 & 4 \end{pmatrix}, \quad d^{A_1 \times A_3} = \begin{pmatrix} 2 & 7 & 6 & 3 \\ 4 & 5 & 2 & 6 \\ 6 & 3 & 5 & 7 \end{pmatrix}, \quad d^{A_2 \times A_3} = \begin{pmatrix} 2 & 5 & 6 & 4 \\ 3 & 4 & 4 & 3 \\ 6 & 5 & 4 & 3 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 12 & 5 & 4 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 7 & 5 & 9 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 3 & 5 & 7 & 6 \end{pmatrix}.$$

Fix $h = 1$. In Step 1 of the algorithm we find that

$$y^{A_1 \times A_2} = \begin{pmatrix} 7 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad y^{A_1 \times A_3} = \begin{pmatrix} 3 & 1 & 2 & 6 \\ 0 & 0 & 5 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}.$$

In Step 2, the inner loop is entered with $a(1) = a(2) = a(3) = 1$, and $q = 0$. Then $z^1_{111} := \min(\tilde{y}^{A_1 \times A_2}_{11}, \tilde{y}^{A_1 \times A_3}_{11}) = \min(7,3) = 3$, $\tilde{y}^{A_1 \times A_2}_{11} := 7 - 3 = 4$, $\tilde{y}^{A_1 \times A_3}_{11} := 0$, and, since $l = 3$, $a(3) := 1 + 1 = 2$, and $q := 3$. Since $q = 3 < 12 = e_{11}$, we enter the inner loop again. Then $z^1_{112} := \min(\tilde{y}^{A_1 \times A_2}_{11}, \tilde{y}^{A_1 \times A_3}_{12}) = \min(4,1) = 1$, $\tilde{y}^{A_1 \times A_2}_{11} := 3$, $\tilde{y}^{A_1 \times A_3}_{12} := 0$, and, since $l = 3$, $a(3) := 3$, and $q := 4$. Again, since $q = 4 < 12 = e_{11}$, we enter the inner loop. Then $z^1_{113} := \min(\tilde{y}^{A_1 \times A_2}_{11}, \tilde{y}^{A_1 \times A_3}_{13}) = \min(3,2) = 2$, $\tilde{y}^{A_1 \times A_2}_{11} := 1$, $\tilde{y}^{A_1 \times A_3}_{13} := 0$, and, since $l = 3$, $a(3) := 4$, and $q := 6$. Since $q = 6 < 12 = e_{11}$, we enter the inner loop. Then $z^1_{114} := \min(\tilde{y}^{A_1 \times A_2}_{11}, \tilde{y}^{A_1 \times A_3}_{14}) = \min(1,6) = 1$, $\tilde{y}^{A_1 \times A_2}_{11} := 0$, $\tilde{y}^{A_1 \times A_3}_{11} := 5$, and, since $l = 2$, $a(2) := 2$, and $q := 7$. Since $q = 7 < 12 = e_{11}$, we enter the inner loop. Then $z^1_{124} := \min(\tilde{y}^{A_1 \times A_2}_{12}, \tilde{y}^{A_1 \times A_3}_{14}) = \min(5,5) = 5$, $\tilde{y}^{A_1 \times A_2}_{12} := \tilde{y}^{A_1 \times A_3}_{14} := 0$, $q := 12$. Since $q = 12 \geqslant 12 = e_{11}$, we leave the inner loop.

Step 2 proceeds by setting $a(1) = 2$, and $a(2) = a(3) = 1$, and by entering the inner loop with $q = 0$. After four iterations of the inner loop we find that $z^1_{233} = 5$. Then, we leave the inner loop, set $a(1) = 3$, and $a(2) = a(3) = 1$, and $q = 0$, and enter the inner loop again. After three iterations of the inner loop we find that $z^1_{332} = 4$, and the algorithm stops. Summarizing, the solution found is z^1 with $z^1_{111} = 3$, $z^1_{112} = 1$, $z^1_{113} = 2$, $z^1_{114} = 1$, $z^1_{124} = 5$, $z^1_{233} = 5$, and $z^1_{332} = 4$, and all other $z^1_a = 0$. This is a feasible solution, as is easily verified.

Notice that, for a given h, the solution constructed by the Single-Hub heuristic depends only on the distance function d and not on the actual costs c_a. We now establish the correctness of the algorithm.

Theorem 2.1. *The solution z^h found by the Single-Hub heuristic is a feasible solution to (kTP).*

Proof. Consider Step 2 of the Single-Hub heuristic. Suppose $a(h) = j$, and $a(r) := i$, for some $r \neq h$, $1 \leqslant i \leqslant n_r$. Since for a specific value of $a(h)$, $a(r)$ is increasing in value, no clusters $a \in A$ with $a(h) = j$ and $a(r) = i$, have been considered before. Hence, let us now focus on the next L consecutive iterations for which $a(h) = j$ and $a(r) = i$ is the case. Thus, in each iteration, some a_t is considered with $a_t(h) = j$, $a_t(r) = i$ for $t = 1,\dots,L$. The value $z^h_{a_t}$ assigned to each cluster $a_t \in A$ is subtracted from $\tilde{y}^{A_r \times A_h}_{ij}$, and finally, in the L th iteration, when the minimum is attained for $\ell = r$, $\tilde{y}^{A_r \times A_h}_{ij}$ is reduced to 0. In other words, the value of $y^{A_r \times A_h}_{ij}$ is distributed over all $z^h_{a_t}$, $t = 1,\dots,L$. Moreover, $a_L(r) := i + 1$, and no clusters $a \in A$ with $a(h) = j$, $a(r) = i$ will be considered in all next iterations. Hence, for each $i \in A_r$, $j \in A_h$, we have

$$\sum_{t=1}^{L} z^h_{a_t} = \sum_{\substack{a\, a(h)=j \\ a(r)=i}} z^h_a = y^{A_r \times A_h}_{ij} \qquad \text{for all } r \in K \setminus \{h\}.$$

Summing over j yields

$$\sum_{j=1}^{n_h} \sum_{\substack{a.a(h)=j \\ a(r)=i}} z^h_a = \sum_{j=1}^{n_h} y^{A_r \times A_h}_{ij} = e_{ri} \qquad \text{for all } r \in K \setminus \{h\}, i \in A_r,$$

where the last equality follows since $y^{A_r \times A_h}$ is a feasible solution to the two-index transportation problem between A_r and A_h.

Similarly, we have $\sum_{i=1}^{n_r} \sum_{\substack{a.a(h)=j \\ a(r)=i}} z^h_a = \sum_{i=1}^{n_r} y^{A_r \times A_h}_{ij} = e_{hj}$ for $j \in A_h$. Therefore, z^h_a is a feasible solution to (kTP). □

Regarding the complexity of the algorithm, notice that one iteration of the inner loop takes O(k) time. The number of iterations of the inner loop is bounded by the sum $\sum_{r \in K \setminus \{h\}} n_r$ of the number of elements in the sets A_r, $(r \neq h)$. Therefore, the complexity of Step 2 equals $n_h \sum_{r \in K \setminus \{h\}} n_r \cdot k$. This is $O(k^2 n^2)$ if $n_r = O(n)$ for all $r \in K$. Since in Step 1, O(k) transportation problems have to be solved, the overall complexity is $O(\sum_{r \in K \setminus \{h\}} (T(n_h, n_r) + n_h n_r k))$, where $T(p,q)$ is the time needed to solve a $p \times q$ transportation problem. With $n_r = O(n)$ for all $r \in K$ and $T(p,q) = O(pq \log(p+q)(pq + (p+q)\log(p+q)))$ (see [10]), we obtain an overall complexity of $O(kn^4 \log n + k^2n^2)$ for the Single-Hub heuristic. Notice that this complexity is polynomial in k and n, in spite of the fact that the number of variables in the formulation of (kTP) in Section 1 is $O(n^k)$.

The *Multiple-Hub* heuristic is derived from the Single-Hub heuristic in the following straightforward way: apply the Single-Hub heuristic for $h = 1, \ldots, k$ and pick the best solution. Its complexity is equal to k times the complexity of the Single-Hub heuristic.

3. A general worst-case analysis

In this section we will establish upper bounds on the ratio between the cost of solutions found by the heuristics and the cost of an optimal solution. Notice that these bounds remain valid when x_a is restricted to be integer for all $a \in A$. This is due to the fact that the solution found by the Single-Hub heuristic is composed from $k - 1$ solutions to ordinary (that is, two-index) transportation problems, using only additions and subtractions. Since the transportation problem has an optimal solution which is integral when all e_{ri} are integral, (see, for instance, [9]), adding the integrality constraint, will only increase the cost of an optimal solution, which implies that the bounds remain valid. Obviously, the problem with integer decision variables is a direct generalization of the multi-index assignment problem with decomposable costs, dealt with in [2]. We will show that their bounds remain valid in this more general setting.

In the sequel of this paper, the superscripts of the length function d are omitted when no confusion is likely to arise. Define, for some h with $1 \leqslant h \leqslant k$, and for each $a \in A$:

$$H_a = \sum_{r \in K \setminus \{h\}} d_{a(r),a(h)},$$

referred to as a *hub*. For a given cost-function f, it may be possible to bound c_a from above in terms of the hub H_a. More precisely, instances arising in practical applications often admit a certain structure which can be captured by introducing a parameter $\alpha_1(k)$, $k \geqslant 2$, such that the following inequality holds for all $a \in A$:

$$c_a \leqslant \alpha_1(k) \cdot H_a$$

To illustrate this, consider the following example.

Example 3.1. Suppose f is the sum-cost function, that is

$$c_a = \sum_{s=2}^{k} \sum_{r<s} d_{a(r),a(s)} \qquad \text{for } a \in A.$$

Bandelt et al. [2] show that, when d satisfies the triangle inequality (see (16)), then

$$c_a \leqslant (k-1) \cdot H_a \text{ for all } a \in A.$$

Thus, $\alpha_1(k) = k - 1$, for $k \geqslant 2$.

Of course, depending on a specific situation, similar analyses can be done for other cost-functions, and other restrictions on the length function.

In a similar way, a parameter $\alpha_2(k)$ is introduced such that the following inequality holds for all $a \in A$:

$$H_a \leqslant \alpha_2(k) \cdot c_a.$$

Obviously, for Example 3.1, $\alpha_2(k) = 1$ for all $k \geqslant 2$. Now, let $c(SH_h)$ denote the cost of the solution found by the Single-Hub heuristic, and let OPT denote the cost of an optimal solution to the k-index transportation problem with decomposable costs.

Theorem 3.1. *$c(SH_h) \leqslant \alpha_1(k)\alpha_2(k)OPT$ for all h.*

Proof. First, we show how to 'decompose' a feasible solution to (kTP) into feasible solutions to $\binom{k}{2}$ two-index transportation problems defined by the sets A_r and A_s with demands e_{ri}, e_{sj} and length function $d_{ij}^{A_r \times A_s}$, $r, s \in K$, $r \neq s$, $i \in A_r$, $j \in A_s$.

Let x_a be any feasible solution to (kTP). We claim that, for any $r, s \in K$, $r \neq s$, the numbers

$$\sum_{\substack{a:a(r)=i,\\ a(s)=j}} x_a, \quad i \in A_r, \ j \in A_s$$

constitute a feasible solution to the 2-index transportation problem between A_r and A_s. Indeed, since for any $r, s \in K$, $r \neq s$,

$$\sum_{j \in A_s} \sum_{\substack{a:a(r)=i,\\ a(s)=j}} x_a = \sum_{a:a(r)=i} x_a = \sum_{a \in A_{ri}} x_a = e_{ri} \qquad \text{for } i \in A_r,$$

and, similarly

$$\sum_{i \in A_r} \sum_{\substack{a:a(s)=j\\ a(r)=i}} x_a = \sum_{a \in A_{sj}} x_a = e_{sj} \qquad \text{for } j \in A_s,$$

the claim is true. In the remainder of the proof x_a, $a \in A$ denotes an optimal solution to (kTP).

Now, we can prove the desired result:

$$c(SH_h) = \sum_{a \in A} c_a z_a^h \tag{1}$$

$$\leqslant \alpha_1(k) \sum_{a \in A} \left(\sum_{r \in K \setminus \{h\}} d_{a(r),a(h)}^{A_r \times A_h} \right) z_a^h \tag{2}$$

$$= \alpha_1(k) \sum_{r\in K\setminus\{h\}} \sum_{i\in A_r} \sum_{j\in A_h} d_{ij}^{A_r\times A_h} \sum_{\substack{a\,a(r)=i,\\ a(h)=j}} z_a^h \tag{3}$$

$$= \alpha_1(k) \sum_{r\in K\setminus\{h\}} \sum_{i\in A_r} \sum_{j\in A_h} d_{ij}^{A_r\times A_h} y_{ij}^{A_r\times A_h} \tag{4}$$

$$\leqslant \alpha_1(k) \sum_{r\in K\setminus\{h\}} \sum_{i\in A_r} \sum_{j\in A_h} d_{ij}^{A_r\times A_h} \sum_{\substack{a:a(r)=i,\\ a(h)=j}} x_a \tag{5}$$

$$= \alpha_1(k) \sum_{a\in A} \left(\sum_{r\in K\setminus\{h\}} d_{a(r),a(h)}^{A_r\times A_h} \right) \cdot x_a \tag{6}$$

$$\leqslant \alpha_1(k)\alpha_2(k) \sum_{a\in A} c_a x_a = \alpha_1(k)\alpha_2(k)\, OPT. \tag{7}$$

Inequalities (2) and (7) hold by definition of $\alpha_1(k)$ and $\alpha_2(k)$, respectively, (3) and (6) are a rearrangement of terms, (4) follows from Theorem 2.1, and (5) follows from the first part of this proof and the fact that $y^{A_r\times A_h}$ is an optimal solution to the transportation problem between A_r and A_h. □

Evidently, the bound $\alpha_1(k)\alpha_2(k)$, $k \geqslant 2$, is also a valid upper bound for the ratio between the cost of the solution found by the Multiple-Hub heuristic and the cost of an optimal solution. In order to be able to derive a possibly better bound for the Multiple-Hub heuristic, we introduce a parameter $\alpha_3(k)$, $k \geqslant 2$, such that the following inequality holds for all $a \in A$:

$$\sum_{s=2}^{k} \sum_{r<s} d_{a(r),a(s)} \leqslant \alpha_3(k) c_a.$$

For example, $\alpha_3(k)$ is easily seen to equal 1 for Example 3.1. With $c(MH)$ denoting the cost of the solution found by the Multiple-Hub heuristic, we have the following theorem:

Theorem 3.2. $$c(MH) \leqslant \frac{2\alpha_1(k)\alpha_3(k)}{k} \cdot OPT.$$

Proof.

$$c(MH) = \min_{h=1,\dots,k} c(SH_h) \tag{8}$$

$$\leqslant \frac{1}{k} \sum_{h=1}^{k} c(SH_h) \tag{9}$$

$$\leqslant \frac{1}{k} \sum_{h=1}^{k} \alpha_1(k) \sum_{a\in A} \left(\sum_{r\in K\setminus\{h\}} d_{a(r),a(h)}^{A_r\times A_h} \right) z_a^h \tag{10}$$

$$= \frac{\alpha_1(k)}{k} \sum_{h=1}^{k} \sum_{r \in K\setminus\{h\}} \sum_{i \in A_r} \sum_{j \in A_h} d_{ij}^{A_r \times A_h} y_{ij}^{A_r \times A_h} \tag{11}$$

$$\leqslant \frac{\alpha_1(k)}{k} \sum_{h=1}^{k} \sum_{r \in K\setminus\{h\}} \sum_{i \in A_r} \sum_{j \in A_h} d_{ij}^{A_r \times A_h} \sum_{\substack{a: a(r)=i,\\ a(h)=j}} x_a \tag{12}$$

$$= \frac{\alpha_1(k)}{k} \sum_{a \in A} \left(\sum_{h=1}^{k} \sum_{r \in K\setminus\{h\}} d_{a(r),a(h)}^{A_r \times A_h} \right) x_a \tag{13}$$

$$= \frac{\alpha_1(k)}{k} \sum_{a \in A} \left(2 \sum_{h=2}^{k} \sum_{r<h} d_{a(r),a(h)}^{A_r \times A_h} \right) x_a \tag{14}$$

$$\leqslant \frac{2\alpha_1(k)\alpha_3(k)}{k} \sum_{a \in A} c_a x_a \leqslant \frac{2\alpha_1(k)\alpha_3(k)}{k} OPT. \tag{15}$$

Inequality (9) is trivial, (10) follows from Theorem 3.1, (11) follows from Theorem 2.1, (12) holds since $y^{A_r \times A_h}$ is an optimal solution to the transportation problem between A_r and A_h, (13) is a rearrangement of terms, (14) follows from the symmetry of the distance function and (15) follows from the definition of $\alpha_3(k)$. □

Finally, notice that $\alpha_2(k) \leqslant \alpha_3(k) \leqslant (k-1)\alpha_2(k)$ for all $k \geqslant 2$. This follows from the fact that $H_a \leqslant \sum_{r<s} d_{a(r),a(s)} \leqslant (k-1)H_a$.

4. Some specific cost functions

To derive meaningful bounds from Theorems 3.1 and 3.2, it is necessary to compute values for the parameters $\alpha_i(k)$, $i = 1,2,3$ introduced in the previous section. In order to compute these values, some restriction on the distance function d is needed. A 'natural' restriction to consider is that d satisfies the triangle inequality

$$d_{a(r),a(s)} \leqslant d_{a(r),a(t)} + d_{a(t),a(s)} \quad \text{for all } r,s,t \in K. \tag{16}$$

Notice that other restrictions on d can also lead to values for $\alpha_i(k)$, $i = 1,2,3$ and, by Theorems 3.1 and 3.2, to worst-case ratios for our heuristics under these alternate restrictions. For instance, in some situations one can assume the following extended triangle (or hypermetric) inequality to hold:

$$d_{a(r),a(s)} \leqslant \max\ \{d_{a(r),a(t)}, d_{a(t),a(s)}\} \quad \text{for all } r,s,t \in K. \tag{17}$$

However, in this section we compute values for $\alpha_i(k)$, $i = 1,2,3$, with respect to specific cost-functions f under the assumption that d satisfies the triangle inequality. We distinguish the following 6 cases:

(i) sum costs: $c_a = \sum_{s=2}^{k} \sum_{r<s} d_{a(r),a(s)}$;

(ii) star costs: $c_a = \min_{\ell \in K} \sum_{r \in K\setminus\{\ell\}} d_{a(r),a(\ell)}$;

(iii) tour costs: $c_a = \min\left\{\sum_{i=1}^{k-1} d_{a(\sigma(i)),a(\sigma(i+1))} + d_{\sigma(k),\sigma(1)}\colon \sigma \text{ is a cyclic permutation of } \{1,\ldots,k\}\right\}$;

(iv) tree costs: $c_a = \min\left\{\sum_{a(i),a(j)\in T} d_{a(i),a(j)} : T \text{ is the edge set of a tree on vertices } a(1),a(2),\ldots,a(k)\right\}$;

(v) diameter costs: $c_a = \max_{r,s} d_{a(r),a(s)}$;

(vi) path costs: $c_a = \min\left\{\sum_{i=1}^{k-1} d_{a(\sigma(i)),a(\sigma(i+1))}\colon \sigma \text{ is a cyclic permutation of } \{1,\ldots,k\}\right\}$.

As mentioned in Section 1, we remark here that proofs in Crama and Spieksma [5] can be used to show that k-index assignment problems with each of these cost-functions is $\mathcal{NP}$-hard for $k \geqslant 3$, even if d satisfies the triangle inequality. Obviously, this holds a fortiori for the multi-index transportation problem with integer variables.

Cases (i)–(iv) are dealt with in [2]. They find, implicitly, the following values for $\alpha_1(k)$, $\alpha_2(k)$ and $\alpha_3(k)$ (see Table 1).

Substituting the appropriate $\alpha_1(k)$, $\alpha_2(k)$ and $\alpha_3(k)$ values into Theorems 3.1 and 3.2 yields the following worst-case bounds for the Single- and Multiple-Hub heuristic with respect to the multi-index transportation problem with decomposable costs.

Corollary 4.1. *For each of the cost-functions* (i)–(iv), $c(SH_h) \leqslant (k-1)\,OPT$ *for all instances satisfying the triangle inequality.*

Corollary 4.2. *For the cost-functions* (i), (ii), $c(MH) \leqslant 2(k-1)/k\,OPT$, *for all instances satisfying the triangle inequality, and for the cost-functions* (iii), (iv), $c(MH) \leqslant \frac{1}{2}k\,OPT$ *if k even and* $c(MH) \leqslant \frac{1}{2}(k-1/k)\,OPT$ *if k odd, for all instances satisfying the triangle inequality.*

Corollaries 4.1 and 4.2 generalize the results in [2] for the multi-index assignment problem. Moreover, examples of problem instances in [2] show that, for the cost-functions (i)–(iv), these worst-case ratios are tight for all $k \geqslant 2$, for both the Single- and Multiple-Hub heuristic. (Obviously, this implies that the bounds, shown here to

Table 1
Values of $\alpha_1(k), \alpha_2(k)$ and $\alpha_3(k)$ for different cost-functions.

	$\alpha_1(k)$	$\alpha_2(k)$	$\alpha_3(k)$
Sum costs	$k-1$	1	1
Star costs	1	$k-1$	$k-1$
Tour costs	2	$\frac{k-1}{2}$	$\frac{1}{8}k^2$ if k even
			$\frac{1}{8}(k+1)(k-1)$ if k odd
Tree costs	1	$k-1$	$\frac{1}{4}k^2$ if k even
			$\frac{1}{4}(k+1)(k-1)$ if k odd

remain valid for the multi-index transportation problem with integer variables, remain tight.)

Let us now consider cost-functions (v) and (vi). In case of the diameter cost function, we can prove the following.

Theorem 4.3. *For the diameter cost-function, $c(SH_h) \leqslant (k-1)\,OPT$, and $c(MH) \leqslant (k-1)\,OPT$, for all instances satisfying the triangle inequality. Moreover, there exist instances with $k = 3$, satisfying the triangle inequality, for which these bounds are tight.*

Proof. $c_a = \max_{r,s} d^{A_r \times A_s}_{a(r),a(s)} \leqslant d^{A_r \times A_h}_{a(r),a(h)} + d^{A_s \times A_h}_{a(s),a(h)} \leqslant H_a$. It follows that $\alpha_1(k) \leqslant 1$, for all $k \geqslant 2$. Also, since for each $r \in K \setminus \{h\}$, we have $d^{A_r \times A_h}_{a(r),a(h)} \leqslant \max_{r,s} d^{A_r \times A_s}_{a(r),a(s)} = c_a$, it follows that $\alpha_2(k) \leqslant k-1$, and $\alpha_3(k) \leqslant \binom{k}{2}, k \geqslant 2$.

Applying Theorems 3.1 and 3.2 yields the desired bounds. These bounds are tight for $k = 3$ as witnessed by the following problem instance of the 3-index assignment problem with diameter costs. Let $A_1 = \{i_1, i_2, \ldots, i_6\}$, $A_2 = \{j_1, \ldots, j_6\}$, $A_3 = \{k_1, \ldots, k_6\}$, and consider the graph depicted in Fig. 1.

A drawn edge in Fig. 1 indicates a distance of 1; any other distance is equal to the length of the shortest path. Further, the demand associated to each node equals 1. The optimal solution is found by setting $x_a = 1$ for $a \in \{(i_1, j_3, k_5), (i_2, j_4, k_6), (i_3, j_6, k_1), (i_4, j_5, k_2), (i_5, j_1, k_4), (i_6, j_2, k_3)\}$ and $x_a = 0$ otherwise. This solution has a total cost of 6. Now, consider the Single-Hub heuristic with $h = 1$. Optimal solutions to the 2 two-index assignment problems defined by A_1 and A_2, and A_1 and A_3, are given by respectively $y^{A_1 \times A_2}_{i_\ell j_\ell} = 1$ and $y^{A_1 \times A_3}_{i_\ell k_\ell} = 1$ for $\ell = 1, \ldots, 6$, and 0 otherwise. Next, in Step 2 of the Single-Hub heuristic, we find the solution $z_a = 1$ for $a = (i_l, j_l, k_l)$, $l = 1, \ldots, 6$, and $z_a = 0$ otherwise, which has total cost equal to 12. The symmetry of this instance implies that indeed the Multiple-Hub heuristic may find a solution with a total cost of 12, as can be verified by the reader. □

In case the cost of each cluster $a \in A$ is given by the shortest Hamiltonian path, we have the following result.

Theorem 4.4. *For the path cost-function,*

$$c(SH_h) \leqslant (2k-4)\,OPT \quad \textit{for } k \geqslant 3$$

and

$$c(MH) \leqslant \begin{cases} (k - 1 - \frac{1}{k-1})\,OPT & \textit{if } k \textit{ even } (k > 2), \\ (k - 1 - \frac{2}{k})\,OPT & \textit{if } k \textit{ odd}, \end{cases}$$

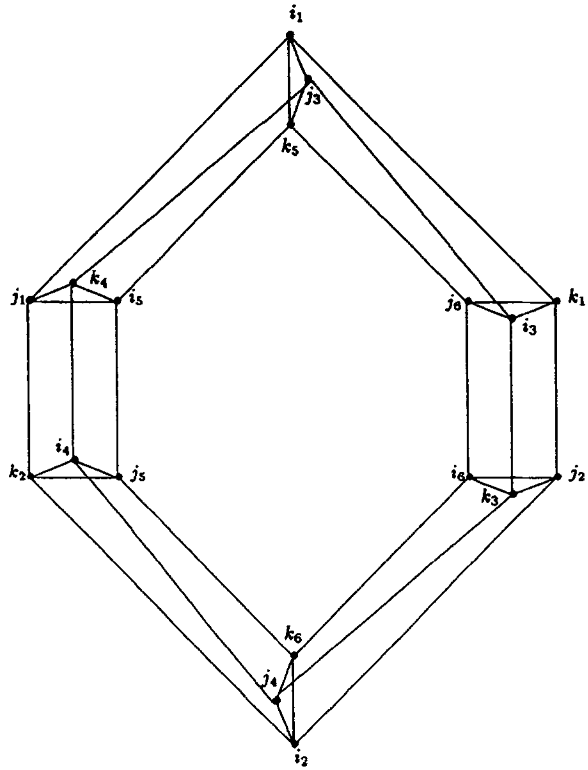

Fig. 1. Worst-case instance for the Single- and Multiple-Hub heuristic for $k = 3$ (diameter costs).

for all instances satisfying the triangle inequality. Moreover, there exist instances with $k = 3$, satisfying the triangle inequality, for which these bounds are tight.

Proof. In order to prove the theorem, we will derive consecutively the values of $\alpha_1(k), \alpha_2(k)$ and $\alpha_3(k)$. Next, we can apply Theorems 3.1 and 3.2. Consider a cluster $a \in A$ with path costs:

$$c_a = \min\left\{\sum_{i=1}^{k-1} d_{a(\sigma(i)),a(\sigma(i+1))} : \sigma \text{ is a cyclic permutation of } \{1,\ldots,k\}\right\}.$$

Using the triangle inequality (16), it is not difficult to show that

$$c_a \leqslant 2 \sum_{j \in K\setminus\{h\}} d_{a(h),a(j)} - d_{a(h),a(r)} - d_{a(h),a(s)} \quad \text{for any } r,s \in K \setminus \{h\}, r \neq s. \tag{18}$$

Now, consider the $k-1$ distances $d_{a(h),a(j)}$, $j \in K \setminus \{h\}$, determining the hub H_a. We order the elements of the cluster a by $j_1, j_2, \ldots, j_{k-1}$ such that $d_{a(h),a(j_i)} \geqslant d_{a(h),a(j_{i+1})}$ for $i = 1,\ldots,k-2$. For the two largest distances in the hub H_a we have

$$d_{a(h),a(j_1)} + d_{a(h),a(j_2)} \geqslant \frac{2}{k-1} H_a. \tag{19}$$

The following argument shows, by contradiction, that (19) is true. Suppose that (19) is not true; then

$$d_{a(h),a(j_i)} \leqslant d_{a(h),a(j_2)} < \frac{1}{k-1} H_a \quad \text{for } i \geqslant 3.$$

But then

$$H_a = \sum_{j \in K \setminus \{h\}} d_{a(h),a(j)} = d_{a(h),a(j_1)} + d_{a(h),a(j_2)} + \sum_{i=3}^{k-1} d_{a(h),a(j_i)} < \frac{2}{k-1} H_a + \frac{k-3}{k-1} H_a$$
$$= H_a,$$

which is a contradiction. Since (18) holds for any $r, s \in K \setminus \{h\}$, we may take in (18), $r = j_1$ and $s = j_2$. Together with (19), we arrive at

$$c_a \leqslant 2(1 - \frac{1}{k-1}) H_a, \tag{20}$$

thus showing that $\alpha_1(k) = 2(1 - 1/(k-1))$.

Regarding $\alpha_2(k)$, since $d_{a(h),a(j)} \leqslant c_a$ for all j, it follows easily that $\alpha_2(k) \leqslant k-1$ for $k \geqslant 2$. Let us now derive an estimate for $\alpha_3(k)$. To do this, we need to refer to the cost of the shortest Hamiltonian *tour* through a cluster, and to its corresponding $\alpha_3(k)$ function. Thus, let c_a^{tour} refer to these shortest tour costs and let $\alpha_3^{\text{tour}}(k)$ refer to the corresponding function (where c_a and $\alpha_3(k)$ are still defined with respect to the shortest Hamiltonian *path* costs). Since the cost of a shortest Hamiltonian tour is smaller or equal to twice the length of a path we derive

$$\sum_{r<s} d_{a(r),a(s)} \leqslant \alpha_3^{\text{tour}}(k) c_a^{\text{tour}}$$
$$\leqslant 2\alpha_3^{\text{tour}}(k) c_a \quad \text{for all } a \in A.$$

Table 1 gives

$$\alpha_3(k) \leqslant 2\alpha_3^{\text{tour}}(k) = \frac{1}{4}k^2 \quad \text{if } k \text{ even,}$$
$$\frac{1}{4}(k+1)(k-1) \quad \text{if } k \text{ odd.}$$

Now, applying Theorems 3.1 and 3.2 to the estimates derived here for $\alpha_1(k)$, $\alpha_2(k)$ and $\alpha_3(k)$ results in the bounds for the Single- and Multiple-Hub heuristic.

In order to show that these bounds are tight for both the Single-Hub and Multiple-Hub heuristic if $k = 3$, consider the following two instances of the 3-index assignment problem with path costs, depicted in Figs 2 and 3 (cf. with [2]).

A drawn edge in Fig. 2 or 3 indicates a distance of 1; any other distance is equal to 2. Further, the demand associated to each node equals 1. Let $A_1 = \{i_1, i_2, i_3\}$, $A_2 = \{j_1, j_2, j_3\}$, $A_3 = \{k_1, k_2, k_3\}$ in both figures. Now, consider the instance for the Single-Hub heuristic, depicted in Fig. 2. The optimal solution is found by setting $x_a = 1$ for $a \in \{(i_1, j_2, k_3), (i_2, j_3, k_1), (i_3, j_1, k_2)\}$, and $x_a = 0$ otherwise. This solution has a total

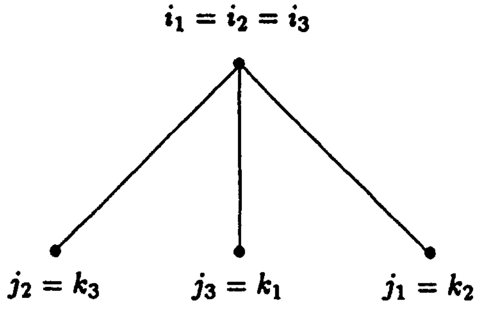

Fig. 2. Worst-case instance for the Single-Hub heuristic for $k = 3$ (path costs).

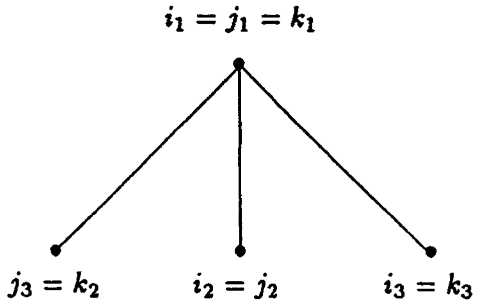

Fig. 3. Worst-case instance for the Multiple-Hub heuristic for $k = 3$ (path costs).

cost of 3. The Single-Hub heuristic with $h = 1$ may find as optimal solutions to the 2 two-index assignment problems defined by A_1 and A_2, and A_1 and A_3, $y_{i_\ell j_\ell}^{A_1 \times A_2} = 1$ and $y_{i_\ell k_\ell}^{A_1 \times A_3} = 1$ for $\ell = 1,2,3$, and 0 otherwise. Next, in Step 2 of the Single-Hub heuristic, we find the solution $z_a = 1$ for $a = (i_l, j_l, k_l)$, $l = 1,2,3$, and $z_a = 0$ otherwise, which has total cost equal to 6, thus achieving the desired ratio for $k = 3$. In fact, it is not difficult to generalize this instance in such a way that the Single-Hub heuristic produces solutions bounded by $(2k - 4)OPT$, showing that the bound proven here is tight for any $k \geqslant 3$, (see [2]).

Let us now consider the Multiple-Hub heuristic and the corresponding instance depicted in Fig. 3. The symmetry of this instance implies that we may restrict ourselves to investigating the performance of the Single-Hub heuristic for this instance. The optimal solution is found by setting $x_a = 1$ for $a \in \{(i_1, j_3, k_2), (i_2, j_2, k_1), (i_3, j_1, k_3), \}$ and $x_a = 0$ otherwise. This solution has a total cost of 3. Now, consider the Single-Hub heuristic with $h = 1$. Optimal solutions to the 2 two-index assignment problems defined by A_1 and A_2, and A_1 and A_3, are given by respectively $y_{i_\ell j_\ell}^{A_1 \times A_2} = 1$ and $y_{i_\ell k_\ell}^{A_1 \times A_3} = 1$ for $\ell = 1,2,3$, and 0 otherwise. Next, in Step 2 of the Single-Hub heuristic, we find the solution $z_a = 1$ for $a = (i_l, j_l, k_l)$, $l = 1,2,3$, and $z_a = 0$ otherwise, which has total cost equal to 4, thus achieving the desired ratio for $k = 3$. □

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